On Minimizing the Maximum Color for the 1-2-3 Conjecture☆

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Abstract

The 1-2-3 Conjecture asserts that, for every connected graph different from K_2, its edges can be labeled with $1, 2, 3$ so that, when coloring each vertex with the sum of its incident labels, no two adjacent vertices get the same color. This conjecture takes place in the more general context of distinguishing labelings, where the goal is to label graphs so that some pairs of their elements are distinguishable relatively to some parameter computed from the labeling.

In this work, we investigate the consequences of labeling graphs as in the 1-2-3 Conjecture when it is further required to make the maximum resulting color as small as possible. In some sense, we aim at producing a number of colors that is as close as possible to the chromatic number of the graph. We first investigate the hardness of determining the minimum maximum color by a labeling for a given graph, which we show is NP-complete in the class of bipartite graphs but polynomial-time solvable in the class of graphs with bounded treewidth. We then provide bounds on the minimum maximum color that can be generated both in the general context, and for particular classes of graphs. Finally, we study how using larger labels permits to reduce the maximum color.

Keywords: edge labelings; proper vertex-colorings; 1-2-3 Conjecture; small vertex colors.

1. Introduction

1.1. Distinguishing labelings and the 1-2-3 Conjecture

This work takes place in the context of *distinguishing labelings*. The general goal is, given a graph G, to *label* some elements (vertices, edges, etc.) of G so that certain pairs of elements (vertices, edges, etc.) can be *distinguished* through parameters computed from the labeling. As can be noted, this problem is quite general, and considering its several parameters leads to many possible labeling variants that could be considered. This topic has actually been intensively studied in the last decades. This is well illustrated by the dynamic survey [5] that has been regularly updated by Gallian over the years, where over 2000 references on the topic, covering more than 200 labeling techniques, are reported.

In this work, we focus on the labeling variant where, given a graph G, we aim at assigning positive integers to the edges so that every two adjacent vertices of G can be distinguished through their "incident sums of labels". Namely, given a k-labeling

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$\ell : E(G) \to \{1, \dots, k\}$ of G, let us define, for every vertex v, the sum $c_\ell(v) = \sum_{w \in N(v)} \ell(vw)$ of the labels of its incident edges. The value $c_\ell(v)$ is called the *color* of the vertex v (induced by the labeling ℓ). A k-labeling is *proper* if $c_\ell(u) \neq c_\ell(v)$ for every edge uv of G. That is, ℓ is a *proper labeling* of G if c_ℓ is a *proper vertex-coloring* of G. Note that proper labelings are sometimes called *neighbor-sum-distinguishing edge-weightings* in the literature. Throughout this work, we will most of the time consider proper labelings and then we will omit the term "proper" unless there is some ambiguity.

Proper labelings have been receiving ingrowing attention since the introduction of the so-called **1-2-3 Conjecture** in 2004 by Karoński, Łuczak and Thomason [7]. Before giving the exact statement of that conjecture, let us first introduce a few additional details. It is easy to see that a connected graph G admits proper labelings if and only if G is different from K_2 (the connected graph with a single edge) [7]. Thus, when dealing with proper labelings, one should focus on graphs with no connected component isomorphic K_2; in the current context, these graphs are called *nice*. Now, for every nice graph G, it makes sense to find the minimum k such that G admits proper k-labelings; this parameter k for G is sometimes denoted by $\chi_\Sigma(G)$.

The 1-2-3 Conjecture states that for every nice graph G, we should have $\chi_\Sigma(G) \leq 3$. In other words, for almost every graph, we should be able to represent one of its proper vertex-colorings by the incident sums inherited from a labeling assigning positive integers with bounded and very small magnitude. A number of results have been exhibited throughout the years as support to the 1-2-3 Conjecture. In what follows, we give a special focus to the such results that connect to our investigations in this paper; for more details, we refer the interested reader to the survey [9] by Seamone, which, although already outdated now, provides pointers towards most of the most important and interesting results on this topic.

The most important result towards the 1-2-3 Conjecture is due to Kalkowski, Karoński and Pfender [6], who proved that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G. It is worth mentioning that graphs G with $\chi_\Sigma(G) = 3$ exist, so the value 3 in the statement of the conjecture cannot be improved down to 2 in general. Note that the 1-2-3 Conjecture is satisfied for nice complete graphs [3], and nice 3-colorable graphs [7], i.e., in particular, for any bipartite graph G, we have $\chi_\Sigma(G) \leq 3$. While the graphs G with $\chi_\Sigma(G) = 1$ can easily be described (these graphs are the *locally irregular ones*, i.e., those without any edge uv with $d(u) = d(v)$, where $d(x)$ denotes the degree of the vertex x), those G with $\chi_\Sigma(G) = 2$ do not admit a "good" characterization, unless P=NP (as first proved by Dudek and Wajc [4]). For some time, an important question was about the existence of such a good characterization for bipartite graphs. It was not until quite recently that a good characterization of the bipartite graphs G with $\chi_\Sigma(G) = 3$ was provided [10]. That result builds upon several sufficient conditions for a bipartite graph G to satisfy $\chi_\Sigma(G) \leq 2$. In particular, this inequality was shown to hold when G has one of its two partite sets of even size [3]. An alternative proof that $\chi_\Sigma(T) \leq 2$ holds for every nice tree T can be found in [3].

1.2. Proper labelings and maximum color

As described above, the 1-2-3 Conjecture states that for almost every graph G we should be able to represent one of its proper vertex-colorings c_ℓ via the sums of labels incident to the vertices inherited from a labeling ℓ assigning positive integers 1, 2 and 3 as labels. One of the downsides of using labels with low magnitude like this, is that we might lose control over the number of vertex colors (i.e., $|\{c_\ell(v) \mid v \in V(G)\}|$) that are generated by a (proper) labeling ℓ. A consequence is that the number of distinct colors defined by c_ℓ might be much larger than $\chi(G)$, the *chromatic number* (i.e., the smallest number of colors in a proper vertex-coloring) of G. This is well illustrated by the case of locally irregular graphs G: as pointed out above, we have $\chi_\Sigma(G) = 1$, but the number of distinct colors obtained by the unique proper 1-labeling of G is the number of distinct degrees over the vertices (which might be arbitrarily larger than $\chi(G)$: consider e.g. the case where G is bipartite).

In a recent work [1], Baudon, Bensmail, Hocquard, Senhaji and Sopena have investigated the trade-off between using more labels and generating a smaller number of colors (i.e., as close as the chromatic number as possible) by a proper labeling. More precisely, for a set of labels L and a graph G, they considered the parameter $\gamma_L(G)$, which is the minimum number of distinct colors defined by c_ℓ that can be generated by any proper L-labeling ℓ of G. In particular, they proved that, for every nice graph G, we have $\gamma_{\mathbb{Z}}(G) = \chi(G)$, unless in the case of a peculiar class of graphs in which case only $\gamma_{\mathbb{Z}}(G) = \chi(G) + 1$ holds. They also proved that $\gamma_{\{1,2\}}(T)$ is of order $\log_2 \Delta(T)$ for any tree T with maximum degree $\Delta(T)$. Finally, they established the NP-hardness of computing $\gamma_{\{1,2\}}(G)$ for a given graph G, even when G is bipartite.

Our investigations in this topic are in the line of those in [1], which are mainly motivated by the fact that, in general, for any graph G the parameter $\gamma_L(G)$ is bounded above by the minimum maximum color that can be achieved over all proper L-labelings of G. In this paper, we consider sets $L = \{1, \dots, k\}$ of consecutive positive labels only. Thus, for a given nice graph G and some $k \geq \chi_\Sigma(G)$, we are interested in the parameter $mS_k(G)$ (where "mS" stands for "maximum sum"), which is the smallest maximum color over the vertices by a proper k-labeling of G. Precisely, given a (proper) labeling ℓ of G, let $mS(G, \ell) = \max_{v \in V(G)} c_\ell(v)$. Then, $mS_k(G) = \min_\ell mS(G, \ell)$ where the minimum is taken over all (proper) k-labelings ℓ of G.

Several aspects behind the general parameter $mS_k(G)$ seem of interest to us, and are precisely related to some of the questions we investigate in the current work. Many of these questions are related to the simple observation that, for every graph G and $k \geq \chi_\Sigma(G)$, we have $mS_k(G) \in \{\Delta(G), \Delta(G)+1, \dots, k\Delta(G)\}$ where $\Delta(G)$ denotes the maximum degree of G (see upcoming Claim 2.2). So our general aim, when labeling G, is to be as close as possible to the lower bound $\Delta(G)$. This leads in particular to the following questions:

1. What is the precise value of $mS_k(G)$?

2. Can we always reach the lower bound? That is, is there always a k such that $mS_k(G) = \Delta(G)$?

3. Assuming Question 2 is wrong, how close to the lower bound can we get? That is, how large can the difference between $\Delta(G)$ and $\min_{k \geq \chi_\Sigma(G)} mS_k(G)$ be?

4. Assuming Question 2 is right, can we always reach the lower bound using weights $1, \dots, \chi_\Sigma(G)$? That is, do we always have $mS_{\chi_\Sigma(G)}(G) = \Delta(G)$?

5. Assuming the lower bound cannot be reached when using labels $1, \dots, k$, can we always get closer to it by using larger labels $1, \dots, k, \dots, k'$? That is, if we have $mS_k(G) > \Delta(G)$ for some k, is there always a $k' > k$ such that $mS_{k'}(G) < mS_k(G)$?

6. Can we always achieve the minimum maximum color using a fixed sets of labels $\{1, \dots, \alpha\}$, regardless of G? That is, is there an absolute α such that if we have $mS_k(G) > mS_{k'}(G)$ for some $k' > k$, then $k' \leq \alpha$?

7. Can using larger weights reduce the minimum maximum color a lot? That is, assuming $mS_k(G) > mS_{k'}(G)$ for some $k < k'$, how large can the difference between $mS_k(G)$ and $mS_{k'}(G)$ be?

1.3. Results in this paper

In this work, we investigate some of the aspects and questions above, providing partial or full answers to some of them. More precisely, this work is organized as follows:

- As a warm up, we start off, in Section 2, by raising general observations on the parameter mS_k, and by providing optimal results for complete graphs and complete bipartite graphs, two classes of graphs for which the parameter χ_Σ is well understood.
- We then consider algorithmic aspects in Section 3. We first establish a negative result in Section 3.1, showing that determining $mS_k(G)$ for any fixed $k \geq 2$ and bipartite graph G as input is NP-complete. We then establish a positive result in Section 3.2, in which we provide a polynomial-time algorithm for determining $mS_k(G)$ for any graph G with bounded treewidth.
- We then investigate bounds on mS_k in Section 4. We focus mainly on general bipartite graphs in Section 4.1. We then focus further on trees in Section 4.2. In particular, we prove that for every nice tree T with maximum degree Δ, the parameter $mS_2(T)$ is one of three possible values: $\Delta, \Delta + 1$ or $\Delta + 2$. In Section 4.3, we investigate, still in trees, how using larger labels can help decreasing the maximum color by a labeling. Finally, in Section 4.4, we prove that, in general, using larger labels can lead to a drastic decrease of the maximum color by a labeling.
- Concluding remarks and perspectives for further work on the topic are gathered in concluding Section 5.

2. Early observations and warm-up results

In this section, we first introduce some easy claims that will be used throughout this work. We then provide first insights into the parameter mS_k by considering classes of graphs for which the value of χ_Σ is fully understood. Namely, we consider complete graphs and complete bipartite graphs.

2.1. Early observations

We start off by exhibiting general bounds on mS_k.

Claim 2.1. *For every nice graph G and $k \geq 2$, we have $mS_k(G) \leq mS_{k-1}(G)$.*

Proof of the claim. This holds since a $(k-1)$-labeling is also a k-labeling. ◇

Claim 2.2. *For every nice graph G and $k \geq 1$, we have $\Delta(G) \leq mS_k(G) \leq k\Delta(G)$.*

Proof of the claim. Consider any k-labeling ℓ of G. The color of any vertex v is

$$c_\ell(v) = \sum_{w \in N(v)} \ell(vw) \leq \sum_{w \in N(v)} k = kd(v) \leq k\Delta(G).$$

Moreover, for any vertex v with degree $\Delta(G)$, we have

$$c_\ell(v) = \sum_{w \in N(v)} \ell(vw) \geq \sum_{w \in N(v)} 1 = \Delta(G),$$

which concludes the proof. ◇

Claim 2.3. *For every locally irregular graph G and $k \geq 1$, we have $mS_k(G) = \Delta(G)$.*

Proof of the claim. This holds since, by definition of a locally irregular graph, assigning label 1 to all edges yields a labeling ℓ with $mS_k(G, \ell) = \Delta(G)$, which is best possible by Claim 2.2. $\diamond$

Claim 2.4. *Let G be a nice graph and ℓ be a labeling of G. If G contains a path (x, u, v, y) with $d(u) = d(v) = 2$, then $\ell(xu) \neq \ell(vy)$. Moreover, if ℓ is a 2-labeling and G contains a path $P = (v_1, \ldots, v_{4q})$ where $d(v_i) = 2$ for all $1 < i < 4q$ and $q > 1$, then $\ell(v_1v_2) \neq \ell(v_{4q-1}v_{4q})$.*

Proof of the claim. The first statement holds because otherwise $c_\ell(u) = c_\ell(v)$. The second statement follows from the same argument. In particular, every two edges of P being at distance 2 cannot be assigned the same label by a proper labeling. $\diamond$

The following situation depicts a context where locally irregular graphs arise.

Claim 2.5. *Let G be a nice graph, and ℓ be a 2-labeling of G. For every edge uv where $d(u) = d(v)$, the number of edges labeled 1 (and similarly 2) incident to u must be different from the number of edges labeled 1 incident to v. In particular, when G is regular, then all edges labeled 1 form a locally irregular graph, and similarly for all edges labeled 2.*

Proof of the claim. The first part of the statement is because otherwise u and v would have the same color. Indeed, for some $x + 2y$ to be equal to some $x' + 2y'$ for $x + y = x' + y'$, we must have $x = x'$ and $y = y'$. The second part of the statement is because every edge uv of G falls into the conditions of the first part of the statement when G is regular. $\diamond$

2.2. First classes of graphs

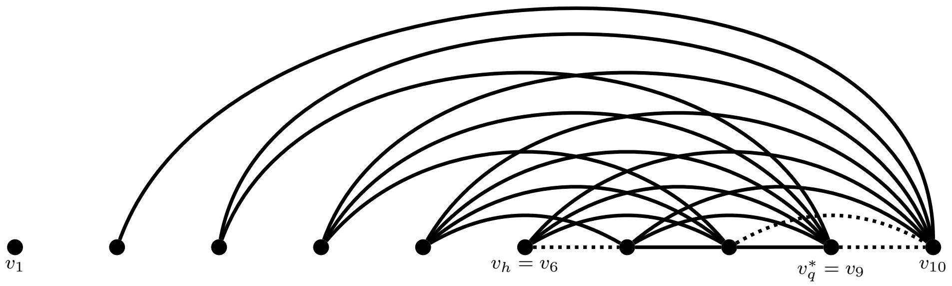
Let us first consider complete graphs K_n with $n \geq 3$, which all verify $\chi_\Sigma(K_n) = 3$. First of all, let us recall that we cannot have $\chi_\Sigma(K_n) \leq 2$, because a 2-labeling of K_n would make the vertex colors to be exactly the distinct values in $\{n-1, \ldots, 2(n-1)\}$, which is impossible since color $n-1$ can only be attained by a vertex incident to 1's only, while color $2(n-1)$ can only be attained by a vertex incident to 2's only [3].

Now, an easy inductive proof of the fact that $\chi_\Sigma(K_n) = 3$ is as follows. Start with a K_3 where all edges receive distinct labels (in $\{1, 2, 3\}$), then add a vertex v_4 adjacent to all other ones via edges labeled 3. Then, for $5 \leq q \leq n$, add a vertex v_q adjacent to all other ones via edges labeled 1 if q is odd, or via edges labeled 3 otherwise. It can easily be checked that a 3-labeling of K_n results whatever is n [3].

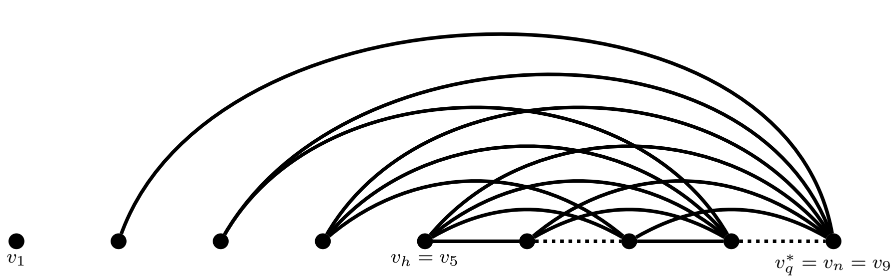
In every resulting labeling ℓ above, we note that $mS(K_n, \ell) = 3(n-1)$ if n is even and $n \geq 4$, while $mS(K_n, \ell) = 3(n-2) + 1$ for $n \geq 5$ otherwise. In the next result, we prove that these values are actually far from $mS_3(K_n)$; in particular, we establish the precise value of this parameter.

Theorem 2.6. *For any $n \geq 3$ and any $k \geq \chi_\Sigma(K_n) = 3$, we have $mS_k(K_n) = 2(n-1) = 2\Delta(K_n)$ if $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$, and $mS_k(K_n) = 2n-1 = 2\Delta(K_n)+1$ otherwise.*

Proof. Let us first prove the lower bounds. In any labeling of K_n, any vertex has color at least $\Delta(K_n) = n-1$ (recall Claim 2.2). Moreover, all vertices must have different colors. Hence, the maximum color must be at least $2(n-1) = 2\Delta(K_n)$. Let us assume that there exists a labeling achieving this lower bound; then the colors of the vertices are $\{n-1, n-2, \ldots, 2(n-1)\}$. Therefore, the sum of their colors must be $S(n) = \sum_{i=0}^{n-1}(n-1+i) =$



(a) $n \equiv 2 \mod 4$



(b) $n \equiv 1 \mod 4$

Figure 1: Examples of optimal labelings of K_n. Full edges are labeled with 2, dotted edges with 3, and all edges that are not represented are labeled with 1.

$\frac{3n(n-1)}{2}$. Moreover, in any graph, the sum of the colors must be even, since every assigned label contributes to the color of exactly two vertices. Since $S(n)$ is even if and only if $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$, we get $mS_k(K_n) \geq 2(n-1)$ if $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$ and $mS_k(K_n) \geq 2n-1$ otherwise. Below, we design optimal labelings in order to establish the equality (see Figure 1 for examples).

In the following, we label all edges in three steps to achieve the lower bounds above. Let $V(K_n) = \{v_1, \ldots, v_n\}$. First, we assign label 1 to every edge. Then all vertices are colored $n-1$. Secondly, we change the labels of the edges in $\{v_iv_j \mid 1 \leq i, j \leq n, i+j \geq n+2\}$ to 2. Then v_1 is incident to no edge labeled 2, vertex v_2 is incident to one edge labeled 2, vertex v_i for $3 \leq i \leq \lfloor (n-1)/2 \rfloor + 1$ is incident to $i-1$ edges labeled with 2, and v_i for $\lfloor (n-1)/2 \rfloor + 2 \leq i \leq n$ is incident to $i-2$ edges labeled with 2. Let $j = \lfloor (n-1)/2 \rfloor + 1$. Note that for every $i \in \{2, 3, \ldots, j, j+2, \ldots, n\}$, v_i is adjacent to one more edge labeled with 2 than v_{i-1}; and that v_j and v_{j+1} are both adjacent to $j-1$ edges labeled with 2 (and $n-j$ edges labeled with 1). So $c_l(v_1) < c_l(v_2) < \cdots < c_l(v_j) = c_l(v_{j+1}) < c_l(v_{j+2}) < \cdots < c_l(v_n)$ and $c_l(v_{i+1}) \leq c_l(v_i) + 1$ for $1 \leq i \leq n$, i.e., all vertices have different colors except v_j and v_{j+1}. Finally, to avoid the conflict between v_j and v_{j+1}, let us increase the label of $v_{j+1}v_{j+2}$ from 2 to 3. This change induces a new conflict between v_{j+2} and v_{j+3}. Then we need to increase the label of $v_{j+3}v_{j+4}$ from 2 to 3 to get rid of this conflict, which creates a new conflict, and so on. Formally, we change the labels of the edges in $\{v_{j+1}v_{j+2}, v_{j+3}v_{j+4}, \ldots, v_{n-1}v_n\}$ to 3 if $n-j$ is even, i.e., if $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$. Otherwise, if $n-j$ is odd and $n \equiv 2 \mod 4$ or $n \equiv 3 \mod 4$, then we change the labels of the edges in $\{v_{j+1}v_{j+2}, v_{j+3}v_{j+4}, \ldots, v_{n-4}v_{n-3}, v_{n-2}v_n, v_{n-1}v_n\}$ to 3. One can check that the resulting labeling is proper, that $mS_k(K_n, \ell) = c_\ell(v_n)$ for any $k \geq 3$, and that $c_\ell(v_n)$

matches the lower bound in all cases. □

In particular, Theorem 2.6 implies that, in complete graphs, it is never necessary to assign a label with value more than $\chi_\Sigma(K_n) = 3$ in order to minimize the maximum color of a vertex. We will see later that it is not always the case.

We now prove a similar result for complete bipartite graphs $K_{n,m}$. We recall that $\chi_\Sigma(K_{n,m})$ is also well understood, since this parameter is 1 if $n \neq m$, and 2 otherwise. In the first situation, this is because $K_{n,m}$ is locally irregular. In the second situation, a 2-labeling of $K_{n,n}$ can be obtained by considering any vertex v, assigning label 2 to all edges incident to v, and assigning label 1 to all other edges [3].

By the previous labeling scheme, we get a 2-labeling ℓ of $K_{n,n}$ where we have $mS(K_{n,n}, \ell) = 2n = 2\Delta(K_{n,n})$. In the next result, we provide 2-labelings with smaller maximum color. Our result is actually optimal.

Theorem 2.7. *For any $1 \leq n < m$ and $k \geq 1$, we have $mS_k(K_{n,m}) = m$. If $n, k \geq 2$, then we have $mS_k(K_{n,n}) = n + 2 = \Delta(K_{n,n}) + 2$ if n is even, and $mS_k(K_{n,n}) = n + 3 = \Delta(K_{n,n}) + 3$ otherwise.*

Proof. If $m > n$, then the result follows from the fact that $K_{n,m}$ is locally irregular and so it admits a 1-labeling (so achieving the lower bound of Claim 2.2).

Now let us assume that $n, k > 1$. Note that $K_{n,n}$ is regular, so it does not admit a 1-labeling. In what follows, we prove that the claimed bounds can be achieved by 2-labelings. Recall that in every 2-labeling of a regular graph, the subgraph induced by the edges labeled 1 (and similarly 2) must be locally irregular (Claim 2.5). In particular, the subgraph induced by the edges labeled with 2 cannot contain an isolated edge uv, which implies that this subgraph must have vertices with degree 2 and, therefore, $mS_2(K_{n,n}) > n + 1$.

Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be the two maximal independent sets of $K_{n,n}$. If n is even, then let us consider the following 2-labeling ℓ. For any $1 \leq i \leq \lfloor n/2 \rfloor$, set $\ell(a_i b_{2i-1}) = \ell(a_i b_{2i}) = 2$, and set $\ell(e) = 1$ for every other edge e. Then, $mS(K_{n,n}, \ell) = n+2$ which is optimal by the previous paragraph. If n is odd, then consider the same labeling as in the previous case, in addition setting $\ell(a_{\lfloor n/2 \rfloor} b_n) = 2$. Then, $mS(K_{n,n}, \ell) = n + 3$. To show that it is optimal, consider any 2-labeling ℓ' with $mS(K_{n,n}, \ell') = n+2$. Let G' be the subgraph induced by the edges with label 2 in ℓ'. Because $mS(K_{n,n}, \ell') = n + 2$, the maximum degree of G' must be 2. Then, G' must be the union of vertex-disjoint paths of length 2 with their vertices of degree 2 being, in G', in a same partite set of $K_{n,n}$, say A. This is because, otherwise, there would be two adjacent vertices, one in A and the other in B, incident to exactly two edges labeled 2 and so G' would not be locally irregular, a contradiction. Hence, A has at least $\lceil n/2 \rceil$ vertices with color n (not adjacent to any edge labeled 2) and B has at least one vertex with color n, contradicting that ℓ' is proper. □

3. Algorithmic complexity

In this section, we investigate the hardness of determining $mS_k(G)$ for a fixed $k \geq 2$ and a graph G given as input. On the negative side, we first prove, in Section 3.1, that this problem is NP-complete for any fixed $k \geq 2$, even when G is a bipartite graph. On the positive side, we then provide, in Section 3.2, a dynamic-programming algorithm establishing that $mS_k(G)$ can be computed in polynomial time when G has bounded treewidth.

3.1. Negative result

Before proceeding with the proof of our main result in this section, Theorem 3.2, we first introduce a few gadgets that will be used for describing the hardness reduction. In that reduction, a key point is that the graph G we will produce has bounded maximum degree $\Delta(G) = 9$. This is why we make some seemingly arbitrary assumptions on some degrees in what follows.

We now introduce *k-gadgets*, for $k \in \{6, \dots, 9\}$. The 9-gadget is nothing but a star with nine leaves, thus with a *center* of degree 9, and rooted in one arbitrary leaf r, while the unique edge incident to r is called the *root edge*. Now, for every $k \in \{6, \dots, 8\}$ such that the $(k+1)$-gadget has been defined, the k-gadget is obtained by starting from an edge ur, and identifying u and the root of each of $k-1$ copies of the $(k+1)$-gadget. The vertex r of the resulting graph is again the *root* of the gadget, while the unique edge incident to the root is the *root edge*. Furthermore, the resulting vertex of degree k is called the *center*.

For a given k-gadget H and a graph G with a vertex v, by *attaching H at v*, we mean adding H to G and identifying v and the root of H.

These gadgets have the following properties:

Observation 3.1. *Let $k \in \{6, \dots, 9\}$. Let G be a graph with maximum degree $\Delta(G) = 9$ obtained from a previous graph by attaching a k-gadget H at one vertex v. Then, in every 2-labeling ℓ of G with $mS(G, \ell) = \Delta(G) = 9$, all edges of H, thus including those incident to v, must be assigned label 1. Furthermore, $c_\ell(v) \neq k$.*

Proof. By construction, every 6-gadget is made of five 7-gadgets, every 7-gadget is made of six 8-gadgets and every 8-gadget is made of seven 9-gadgets. Let us consider a 2-labeling ℓ of G with $mS(G, \ell) = 9$. Let $k \in \{6, \dots, 9\}$ and let H be any k-gadget pending at some vertex v in G. Assume first $k = 9$. Since the center c of any 9-gadget in H has degree 9, all its incident edges must be labeled with 1 so that $c_\ell(c) \leq 9$. Actually, $c_\ell(c) = 9$. Now, let us assume that $k < 9$. By construction, the center c of any 8-gadget in H has degree 8, and seven of its incident edges go to the center of a 9-gadget, and are thus labeled 1 (by the previous paragraph). Furthermore, c is incident to the center of a 9-gadget, so, by the previous paragraph, we must have $c_\ell(c) < 9$. Note that if the remaining edge incident to c is labeled 2, then c would have color 9, a conflict. So it must be labeled 1, in which case $c_\ell(c) = 8$. These arguments generalize as follows. Assume we have shown that all edges of all $(k+1)$-gadgets in H must be labeled 1, and that this forces the color of their center to be $k+1$. Now consider the center c of a k-gadget in H. By construction, $k-1$ of its incident edges go to the center of a $(k+1)$-gadget, and are thus labeled 1. Furthermore, c is adjacent to a vertex with color $k+1$. Now, if the remaining edge incident to c was labeled 2, then the color of c would be $k+1$, a conflict. So it must be labeled 1, and $c_\ell(c) = k$.

Now, v is adjacent, in G, to the center of a k-gadget, and the root edge of that gadget must be labeled 1. Furthermore, the center of that gadget has color k. Therefore, $c_\ell(v) \neq k$. □

We are now ready for the proof of the main theorem of this section.

Theorem 3.2. *Let $k \geq 2$. The problem that takes a bipartite graph G with maximum degree 9 as input and asks whether $mS_k(G) = 9$ is* NP*-complete.*

Proof. Since the NPness of the problem is obvious, we focus on proving its NP-hardness. The proof is done by reduction from Cubic Monotone 1-in-3 SAT, which is NP-hard, see [8]. In this problem, we are given a 3CNF formula F with positive variables only, each of which appears in exactly three clauses, and each clause contains exactly three distinct variables. The question is whether F can be *1-in-3 satisfied*, meaning whether there is a *1-in-3 truth assignment* of F, i.e., a truth assignment to the variables such that every clause has exactly one true variable.

Let us first consider the case $k = 2$. We construct a bipartite graph G with maximum degree $\Delta(G) = 9$, such that F can be 1-in-3 satisfied if and only if $mS_2(G) = \Delta(G)$. The construction is as follows. Start from G being the bipartite graph with bipartition $V \cup C$ modelling the structure of F. That is, for every variable x_i of F there is a *variable vertex* v_i in V, for every clause C_j of F there is a *clause vertex* c_j in C, and for every variable x_i belonging to clause C_j of F we have the *formula edge* v_ic_j in G. Since in every clause of F all variables are distinct, and every variable appears in exactly three clauses, the graph G is actually cubic.

We achieve the construction of G by attaching a 6-gadget and a 7-gadget at every variable vertex v_i, and attaching a 6-gadget, 8-gadget and 9-gadget at every clause vertex c_j. Clearly, the construction of G is achieved in polynomial time. Note that all variable vertices of G have degree 5, while all clause vertices have degree 6. Then, the maximum degree of G is 9, due to the 9-gadgets, and Observation 3.1 applies to G and its k-gadgets.

Therefore, in every 2-labeling ℓ of G with $mS(G, \ell) = 9$, all edges of the k-gadgets are labeled 1, which implies that the variable vertices have color at least 5 and at most 8 (since they have degree 5 and at least two of their incident edges must be labeled 1) different from 6 and 7, and the clause vertices have sum at least 6 (and at most 9) different from 6, 8 and 9. This in turn implies that the three formula edges incident to a variable vertex v_i cannot be labeled so that the sum of their labels is 4 (the color of v_i would be 6) or 5 (the color of v_i would be 7). So the sum of these three edge labels must be either 3 (all formula edges incident to v_i are labeled with 1), in which case v_i has color 5, or 6 (all formula edges incident to v_i are labeled with 2), in which case v_i has color 8. Regarding a clause vertex c_j, its three incident formula edges cannot have labels summing up to 3 (the color of c_j would be 6), 5 (the color of c_j would be 8) or 6 (the color of c_j would be 9). Thus, the sum of these three edge labels must be 4, in which case c_j has color 7. This implies that a variable vertex and a clause vertex cannot be involved in a conflict. Also, for every clause vertex, exactly one of its incident formula edges must be labeled 2, while the other two must be labeled 1.

The equivalence between finding a 1-in-3 truth assignment ϕ to the variables of F and a 2-labeling ℓ of G with $mS(G, \ell) = 9$ now follows from the following arguments. We regard the fact that a formula edge v_ic_j of G is assigned label 2 (resp. 1) by ℓ as having variable x_i of F bringing truth value *true* (resp. *false*) to clause C_j by ϕ. The fact that, in G, all three formula edges incident to a variable vertex v_i must all be labeled 1 or all be labeled 2 by ℓ depicts the fact that, by ϕ, every variable x_i brings the same truth value to the three clauses containing it. The fact that, in G, for every clause vertex c_j, one of its incident formula edges must be labeled 2 by ℓ while the other two must be labeled 1 depicts the fact that, here, a clause C_j is regarded satisfied by ϕ only when it has exactly one true variable.

Now, we extend the above result to any $k \geq 2$. We remark that with slight modifications of our k-gadgets with $k \in \{6, 7\}$, the proof of Theorem 3.2 would go the same way when considering k-labelings with any $k \geq 2$. That is, in the construction of the 7-gadget, replace one of the 8-gadgets attached to the center by a 9-gadget. In the construction of the 6-gadget, replace one of the 7-gadgets attached to the center by an 8-gadget, and another of the 7-gadgets by a 9-gadget. This way, it can be noted that Observation 3.1 remains true for 6-gadgets, 7-gadgets, 8-gadgets and 9-gadgets even when considering k-labelings with $k \geq 2$.

Now, consider the reduction above with some $k > 2$ and these modified gadgets. Again, the color of a clause vertex c_j is at least 6 by a k-labeling, due to its degree. Furthermore, c_j cannot have color 6, 8 or 9 due to the centers of some gadgets attached to it. We note then that if a formula edge incident to c_j was labeled at least 3, then c_j would have color at least 8, which would thus either raise a conflict or make c_j have color more than 9. Thus, no edge of G should be assigned a label more than 2 if we want to get a k-labeling ℓ of G with $mS(G, \ell) \leq 9$. In other words, $mS_2(G) = \Delta(G) = 9$ if and only if F is 1-in-3 satisfiable, in which case we have $mS_k(G) = mS_2(G)$ for every $k \geq 2$. $\square$

It is worth emphasizing that the hardness of the problem established in Theorem 3.2 is not a consequence of the low maximum degree assumption. Indeed, it can be noted that the reduction in the proof can easily be modified to produce bipartite graphs with arbitrarily large maximum degree Δ. In particular, for any fixed Δ, we can quite similarly come up with slightly modified k-gadgets for any $k \in \{\Delta, \Delta - 1, \dots, 2\}$, which can then be used in the reduction the exact same way, to get reduced graphs with maximum degree Δ.

3.2. Positive result

In this section, we show that, for any fixed k, the parameter $mS_k(G)$ (and $\chi_\Sigma(G)$) can be computed in polynomial time in the class of graphs with bounded treewidth.

Let $G = (V, E)$ be any undirected connected simple graph (not reduced to one vertex). A *tree-decomposition* $(T, \mathcal{X})$ of G consists of a tree $T = (V(T), E(T))$ and a set $\mathcal{X} = (X_t)_{t \in V(T)}$ of subsets of V (i.e., $X_t \subseteq V$ for every $t \in V(T)$) satisfying the following two properties:

- for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $\{u, v\} \subseteq X_t$, and
- for every $v \in V$, $\{t \in V(T) \mid v \in X_t\}$ induces a subtree of T.

The *width* of $(T, \mathcal{X})$ equals $\max_{t \in V(T)} |X_t| - 1$ and the *treewidth* $tw(G)$ of G is the minimum width among all tree-decompositions of G. Abusing notations, we will often identify a vertex $t \in V(T)$ with the corresponding set $X_t \in \mathcal{X}$. The sets in $\mathcal{X}$ are also called *bags*.

A tree-decomposition $(T, \mathcal{X})$ is *nice* [2] if T is rooted and every node $t \in V(T)$ is of one of the following four types:

Leaf node: t is a leaf of T and $|X_t| = 1$;

Introduce node: t has a unique child t' and there exists $v \in V$ such that $X_t = X_{t'} \cup \{v\}$;

Forget node: t has a unique child t' and there exists $v \in V$ such that $X_{t'} = X_t \cup \{v\}$;

Join node: t has exactly two children t', t'' and $X_t = X_{t'} = X_{t''}$.

It is well known that any graph G admits a nice tree-decomposition $(T, \mathcal{X})$ rooted insome $r \in V(T)$ with width $tw(G)$ and $|V(T)| = O(|V|)$, and the root bag X_r verifies $X_r = \emptyset$ [2].

Given a rooted tree-decomposition $(T, \mathcal{X})$ and $t \in V(T)$, let T_t denote the subtree of T induced by t and its descendants, and let G_t be the subgraph of G induced by $\bigcup_{t' \in V(T_t)} X_{t'}$. A *partial k-labeling* for G_t consists of two functions $(\ell : E(G_t) \to \{1, \dots, k\}, c : V(G_t) \to \mathbb{N})$ such that c is a proper coloring of G_t, we have $c(v) = \sum\limits_{e \in E(G_t), v \in e} \ell(e)$ for every vertex

$v \in V(G_t) \setminus X_t$, and $c(v) \geq \sum_{e \in E(G_t), v \in e} \ell(e)$ for every vertex $v \in X_t$. Since by the properties of tree-decomposition, X_t separates $G_t - X_t$ from $G - V(G_t)$, i.e., any path from $G_t - X_t$ to $G - V(G_t)$ intersects X_t (in particular there are no edges between a vertex in $G_t - X_t$ and a vertex in $G - V(G_t)$), every k-labeling ℓ of G induces a partial k-labeling $(\ell_{|E(G_t)}, c_{\ell\ |V(G_t)})$ for G_t (where $f_{|X}$ means the function $f : Y \to Z$ restricted to the set $X \subseteq Y$).

Remark that if the root r of T is such that $X_r = \emptyset$, then any partial k-labeling of $G_r = G$ is a k-labeling of G.

Theorem 3.3. *Let $k \geq 2$ and $tw \geq 1$ be two fixed integers. Given an n-node graph G and an integer s as inputs, the problem of deciding whether $mS_k(G) \leq s$ can be solved in polynomial time in the class of graphs G with treewidth at most tw (and in linear time if the maximum degree is bounded).*

Proof. Let $(T, \mathcal{X})$ be a nice tree-decomposition of an n-node graph G with width $tw(G)$ and $|V(T)| = O(n)$, and the root r of T is such that $X_r = \emptyset$. Let Δ be the maximum degree of G. Let $t \in V(T)$, $X_t = \{v_1, \ldots, v_w\}$ ($w \leq tw(G)+1$) and $\{e_1, \ldots, e_q\}$ be the set of the edges induced by X_t in G ($q = O(tw^2)$). Let $L = \{l_1, \ldots, l_q\} \subset \{1, \ldots, k\}^q$, $FC = \{f_1, \ldots, f_w\} \subset \{1, \ldots, k\Delta\}^w$ (FC stands for "final colors") and $CB = \{b_1, \ldots, b_w\} \subset \{0, 1, \ldots, k\Delta\}^w$ (CB stands for "colors from below"). Let us set $\alpha_t(L, FC, CB) = \min_{(\ell,c)} \max_{v \in V(G_t)} c(v)$ where the minimum is taken over all partial k-labeling (ℓ, c) of G_t such that, for any $1 \leq i \leq q$, we have $\ell(e_i) = l_i$ (ℓ agrees with L on the edges in X_t), for any $1 \leq i \leq w$, we have $c(v_i) = f_i$ (c agrees with FC on the vertices in X_t), and, for any $1 \leq i \leq w$, we have $b_i = \sum_{x \in N(v_i) \cap (V(G_t) \setminus X_t)} \ell(v_i x)$ (b_i represents the contribution to the color of v_i from the edges "below" X_t, i.e., between v_i and vertices in $G_t - X_t$). Note that if such a partial labeling exists, then we must have, for every $1 \leq i \leq w$,

$$\sum_{e \in E(G_t), v_i \in e} \ell(e) = b_i + \sum_{1 \leq j \leq q, v_i \in e_j} l_j \leq c(v_i) = f_i.$$

Moreover, let us set $\alpha_t(L, FC, CB) = \infty$ if no such partial k-labeling exists. Finally, let us set

$$Table(t) = ((L, FC, CB, \alpha_t(L, FC, CB)))_{L \subset \{1,\ldots,k\}^q, FC \subset \{1,\ldots,k\Delta\}^w, CB \subset \{0,1,\ldots,k\Delta\}^w}.$$

Note that

$$|Table(t)| = O(k^{(tw(G)+1)^2}(k\Delta + 1)^{2tw(G)+2})$$

since $q \leq \binom{tw(G)+1}{2} = O((tw(G)+1)^2)$ and $w \leq tw(G) + 1$.

We now present a dynamic-programming algorithm that computes $Table(t)$ for all $t \in V(T)$, bottom-up, from the leaves to the root r of T. By the remark preceding the theorem, we get that if the root r of T is such that $X_r = \emptyset$, then we get the result since $Table(r)$ contains the unique value $((\emptyset, \emptyset, \emptyset), mS_k(G))$ (which may be ∞ if and only if $\chi_\Sigma(G) < k$). There are four cases depending on the type of t.

- For every leaf node $t \in V(T)$ of T, $Table(t)$ is defined by setting $\alpha_t(L, FC, CB) = i$ if $(L, FC, CB) = (\emptyset, \{i\}, \{0\})$, and $\alpha_t(L, FC, CB) = \infty$ otherwise.

- Let $t \in V(T)$ be an introduce node, t' be its (unique) child, and let $X_t \setminus X_{t'} = \{v\}$. Moreover, let $|X_{t'}| = w$, $X_{t'} = \{v_1, \ldots, v_w\}$, $\{e_1, \ldots, e_q\}$ be the set of edges induced by $X_{t'}$, and let $\{e_{q+1}, \ldots, e_{q+h}\}$ be the set of edges between v and the vertices in X_t

(w.l.o.g., we may assume that $e_{j+q} = vv_j$ for all $1 \leq j \leq h$). Note that $N(v) \cap V(G_t) \subseteq X_t$ by the properties of tree-decompositions.

For every $L = \{l_1, \ldots, l_{q+h}\} \subset \{1, \ldots, k\}^{q+h}$, $FC = \{f_1, \ldots, f_{w+1}\} \subset \{1, \ldots, k\Delta\}^{w+1}$, $CB = \{b_1, \ldots, b_{w+1}\} \subset \{0, 1, \ldots, k\Delta\}^{w+1}$, let $\alpha' = \alpha_{t'}(\{l_1, \ldots, l_q\}, \{f_1, \ldots, f_w\}, \{b_1, \ldots, b_w\})$ (by induction, this value has been computed and can be found in $Table(t')$).

Now, set $\alpha_t(L, FC, CB) = \max\{f_{w+1}, \alpha'\}$ if and only if all the following conditions (which will ensure that a partial labeling exists) are satisfied and set $\alpha_t(L, FC, CB) = \infty$ otherwise.

- $\{f_1, \ldots, f_{w+1}\}$ induces a proper coloring of X_t (in particular, $f_j \neq f_{w+1}$ for all $1 \leq j \leq h$).
- $b_{w+1} = 0$ (since there are no edges between v and $G_t - X_t$, v cannot have some contribution to its color coming from "below").
- $\sum_{q+1 \leq j \leq q+h} l_j \leq f_{w+1}$ (the current color of v, obtained from the labels of the edges $e_{q+1}, \ldots, e_{q+h}$ (in X_t), cannot exceed its final color f_{w+1}).
- for every $1 \leq j \leq h$, $b_j + \sum_{1 \leq i \leq q+h, v_j \in e_i} l_i \leq f_j$ (adding an edge $vv_j = e_{q+j}$ with label l_{q+j} does not make the vertex v_j to have a color larger than its final color f_j).

• Let $t \in V(T)$ be a forget node, t' be its (unique) child, and let $X_{t'} \setminus X_t = \{v\}$. Moreover, let $|X_t| = w$, $X_t = \{v_1, \ldots, v_w\}$, $\{e_1, \ldots, e_q\}$ be the set of edges induced by X_t, and let $\{e_{q+1}, \ldots, e_{q+h}\}$ be the set of edges between v and the vertices in X_t (w.l.o.g., we may assume that $e_{j+q} = vv_j$ for all $1 \leq j \leq h$). Note that $N[v] \subseteq V(G_{t'})$ by the properties of a tree-decomposition.

For every $L = \{l_1, \ldots, l_q\} \subset \{1, \ldots, k\}^q, FC = \{f_1, \ldots, f_w\} \subset \{1, \ldots, k\Delta\}^w, CB = \{b_1, \ldots, b_w\} \subset \{0, 1, \ldots, k\Delta\}^w$, a tuple (L', FC', CB') is called a *valid extension* of (L, FC, CB) if and only if

extension: $L' = \{l_1, \ldots, l_{q+h}\} \subset \{1, \ldots, k\}^{q+h}, FC' = \{f_1, \ldots, f_{w+1}\} \subset \{1, \ldots, k\Delta\}^{w+1}$, $CB' = \{b_1, \ldots, b_{w+1}\} \subset \{0, 1, \ldots, k\Delta\}^{w+1}$ (i.e., L' coincides with L on its first q values, FC' coincides with FC on its first w values, and CB' coincides with CB on its first w values)

valid: $f_{w+1} = b_{w+1} + \sum_{1 \leq j \leq h} l_{q+j}$ (since v is "forgotten", it will never receive more contribution to its color, so we must ensure that its current color, received from the edges "below" and from the edges in the bag, equals its final color).

Set $\alpha_t(L, FC, CB) = \min_{(L', FC', CB') \texttt{ valid extension of } (L, FC, CB)} \alpha_{t'}(L', FC', CB')$ and set $\alpha_t(L, FC, CB) = \infty$ if no such valid extension exists.

• Let $t \in V(T)$ be a join node, t' and t'' be its two children, $X_t = X_{t'} = X_{t''} = \{v_1, \ldots, v_w\}$, and let $\{e_1, \ldots, e_q\}$ be the set of edges induced by X_t.

Let $L = \{l_1, \ldots, l_q\} \subset \{1, \ldots, k\}^q, FC = \{f_1, \ldots, f_w\} \subset \{1, \ldots, k\Delta\}^w, CB = \{b_1, \ldots, b_w\} \subset \{0, 1, \ldots, k\Delta\}^w$.

If (and only if), for all $1 \leq i \leq w$, we have $f_i \geq b_i + \sum_{1 \leq j \leq q, v_i \in e_j} l_j$ (we consider first all tuples (L, FC, CB) that ensure that the current color of every vertex is not larger than its final color), then let $\mathcal{K}$ be the set of tuples $(k_1, \ldots, k_w)$ such that $0 \leq k_i \leq b_i$ for all $1 \leq i \leq w$. For any $P \in \mathcal{K}$, let $CB - P = (b_1 - k_1, \ldots, b_w - k_w)$. Then, let us set $\alpha(L, FC, CB) = \min_{P \in \mathcal{K}} \max\{\alpha_{t'}(L, FC, P); \alpha_{t''}(L, FC, CB - P)\}$. Intuitively, for every $1 \leq i \leq w$ and given $P = (k_1, \ldots, k_w) \in \mathcal{K}$, the value k_i (resp., $b_i - k_i$) represents the contribution of the edges below $X_{t'}$ (resp., below $X_{t''}$) to the color of v_i.

For every other tuple (L, FC, CB), set $\alpha_t(L, FC, CB) = \infty$.

The correctness of the algorithm (i.e., showing that, for every $t \in V(T)$ and every tuple (L, FC, BW), the tuple $((L, FC, CB), \alpha_t(L, FC, CB))$ computed by the algorithm satisfies the definition given at the beginning of the proof) can be seen true by inductive arguments.

The biggest time complexity occurs in the case of a join node, where $O(k^{(tw(G)+1)^2}(k\Delta + 1)^{2tw(G)+2})$ tuples (L, FC, CB) must be considered and, for each of them, $O((k\Delta+1)^{tw(G)+1})$ tuples $(k_1, \ldots, k_w)$ must be checked. Since $|V(T)| = O(n)$, the overall complexity is $O(nk^{(tw(G)+1)^2}(k\Delta + 1)^{3tw(G)+3})$.

Note that an optimal labeling (i.e., achieving $mS_k(G)$) can also be obtained in polynomial time by a second bottom-up traversal of the tree-decomposition. $\square$

A nice consequence of Theorem 3.3 is that it also provides a polynomial-time algorithm for deciding whether $\chi_\Sigma(G) \leq k$ holds for a given graph G with bounded treewidth. This is, in particular, a consequence of the fact that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G [6]. We are not aware of any such result in the literature.

Corollary 3.4. *The problem of deciding $\chi_\Sigma(G)$ can be solved in polynomial time in the class of graphs G with bounded treewidth.*

4. Bounds on mS_k for some graph classes

In this section, we establish bounds on $mS_k(G)$ for particular classes of graphs. More precisely, we first focus on general bipartite graphs, in Section 4.1, before narrowing down our concern to trees, in Section 4.2. In the rest of the section, we then investigate the effects on our bounds of using larger edge labels. More precisely, we focus on the following questions:

1. Assuming $mS_k(G) = x$ for some k and graph G, what is the smallest $k' > k$ (if any) such that $mS_{k'}(G) < x$? In particular, are there situations where we need much larger labels in order to decrease the maximum color?

2. Assuming $mS_k(G) < mS_{k'}(G)$ for some $k < k'$, how large can the difference between $mS_k(G)$ and $mS_{k'}(G)$ be? In particular, can using larger labels decrease the maximum color a lot?

Regarding the first question, we exhibit, in Section 4.3, trees T for which the smallest k' such that $mS_{k'}(T) < mS_k(T)$ is arbitrarily larger than k. Regarding the second question, we exhibit, in Section 4.4, graphs G for which $mS_2(G) = 2\Delta(G)$ and $mS_3(G) = \Delta(G)$. Hence, using larger labels can make the maximum color drop from the largest possible color to the smallest possible color (recall the bounds in Claim 2.2).

4.1. Bipartite graphs

Since all nice bipartite graphs verify the 1-2-3 Conjecture [7], they can be classified into three classes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, where, for $i = 1, 2, 3$, class $\mathcal{B}_i$ contains the bipartite graphs G with $\chi_\Sigma(G) = i$. Recall that the graphs of $\mathcal{B}_1$ are exactly the bipartite graphs G that are locally irregular, which verify $\chi_\Sigma(G) = 1$, and we thus have $mS_k(G) = \Delta(G)$ for every $k \geq 1$ (due to Claim 2.1), which is best possible by Claim 2.3. Throughout this section, we investigate bounds on the parameter mS_k for the graphs of $\mathcal{B}_2$ and $\mathcal{B}_3$. Recall that [10] provides a full characterization of the graphs of $\mathcal{B}_3$ (see Section 4.1.2).

4.1.1. Graphs of $\mathcal{B}_2$

The graphs of $\mathcal{B}_2$ are those nice bipartite graphs that are neither locally irregular ($\mathcal{B}_1$) nor odd multi-cacti ($\mathcal{B}_3$, see Section 4.1.2 for more details). Since the graphs of $\mathcal{B}_1$ and $\mathcal{B}_3$ are easy to recognize, so are those of $\mathcal{B}_2$. But the structure of the graphs in $\mathcal{B}_2$ is rather general, which makes it difficult to come up with general properties of mS_2 for these graphs. For instance, recall that $\mathcal{B}_2$ includes all non-locally irregular bipartite graphs with one partite set of even size (see [3]).

Recall that, for a graph G of $\mathcal{B}_2$ with maximum degree Δ, we have $\Delta \leq mS_2(G) \leq 2\Delta$ by Claim 2.2. However, Theorem 3.2 shows that determining $mS_2(G)$ for a bipartite graph $G \in \mathcal{B}_2$ is NP-complete (to see that the graphs we construct in the reduction indeed are in $\mathcal{B}_2$, note that they have minimum degree 1, which is a sufficient condition for being neither in $\mathcal{B}_1$ nor in $\mathcal{B}_3$, see upcoming Section 4.1.2). Still, in the next two results, we exhibit families of arbitrarily large graphs of $\mathcal{B}_2$ with "large" value of mS_2. For small values of Δ, we even provide families of arbitrarily large graphs for which mS_2 attains the upper bound 2Δ.

Proposition 4.1. *For any $n_0 \geq 6$ and $\Delta \in \{2, 3\}$, there exist $n \geq n_0$ and a Δ-regular bipartite n-node graph $G \in \mathcal{B}_2$ such that $mS_2(G) = 2\Delta$.*

Proof. First, let $\Delta = 2$. For any $n \geq 4$, the n-node cycle C_n must have, by any 2-labeling, some vertex incident to two edges labeled with 2. Indeed, all edges can clearly not be labeled all with 1. So, let uv be an edge labeled with 2; since the colors of u and v must differ, exactly one of them must have its two incident edges labeled with 2. Hence, for any $n \geq 4$, we have $mS_2(C_n) \geq 4 = 2\Delta = \Delta + 2$ (the equality comes from Claim 2.2). Since some bipartite cycles are indeed in $\mathcal{B}_2$ (see the definition of $\mathcal{B}_3$ in Section 4.1.2), this proves the claim for $\Delta = 2$.

Now, let $\Delta = 3$ and $n = 4k+2$ for some $k \in \mathbb{N}^*$. Let G be obtained from the cycle C_n of size n by adding edges between opposite vertices. That is, start from the cycle $(v_1, \dots, v_n)$ and, for all $1 \leq i \leq 2k+1$, add the edge $v_i v_{i+2k+1}$. Since $G \notin \mathcal{B}_3$ (in particular because G has minimum degree 3, see next Section 4.1.2) and is not locally irregular, $\chi_\Sigma(G) = 2$ (as proved in [10]). Since G is 3-regular, $mS_2(G) \leq 6 = 2\Delta = \Delta + 3$ (by Claim 2.2). Let us prove it is an equality. For purpose of contradiction, let us assume that there exists a 2-labeling ℓ of G such that $mS(G, \ell) < 6$.

We first claim that there are no edges uv such that $c_\ell(u) = 3$ and $c_\ell(v) = 5$. Indeed, let v be a vertex with $c_\ell(v) = 5$ (if any). Let a, b, c be its neighbors. Since $c_\ell(v) = 5$, we must have $\ell(va) = 1, \ell(vb) = 2$ and $\ell(vc) = 2$. Hence, $c_\ell(b) = c_\ell(c) = 4$ (these colors are at least 4 because they have an incident edge labeled 2 and must be different from $c_\ell(v)$, and at most 5 by the hypothesis on ℓ). For purpose of contradiction, let us assume that $c_\ell(a) = 3$. Let x be the common neighbor of c and a. Then, $c_\ell(x) \neq c_\ell(a) = 3$ and $c_\ell(x) \neq c_\ell(c) = 4$. Hence, $c_\ell(x) = 5$ and x must be incident to exactly two edges labeled with 2. But $\ell(ax) = 1$ since $c_\ell(a) = 3$. Hence, $\ell(cx) = 2$, contradicting the fact that $c_\ell(c) = 4$.

Let (A, B) be the bipartition of G. Let v be any vertex such that $c_\ell(v) \in \{3, 5\}$ (there clearly must exist such a vertex) and, w.l.o.g., say $v \in A$. By doing a BFS from v and using the fact that there are no edges uv with $c_\ell(u) = 3$ and $c_\ell(v) = 5$ (and that two adjacent vertices cannot have the same color), we deduce that, for all $w \in V(G)$, $c_\ell(w) = 4$ if and only if $w \in B$.

Therefore, $\sum_{w\in B} c_\ell(w) = 4|B| = 4(2k + 1)$ is even. Moreover, let x be the number of vertices colored 5. Then $\sum_{w\in A} c_\ell(w) = 5x + 3(|A| - x)$. Since $|A|$ is odd, then $\sum_{w\in A} c_\ell(w)$ is odd. However, $\sum_{w\in B} c_\ell(w) = \sum_{w\in A} c_\ell(w) = \sum_{e\in E(G)} \ell(e)$, which contradicts the existence of ℓ. □

Proposition 4.2. *For every $\Delta \geq 2$ and $k \geq 2$, there exists a bipartite graph $G \in \mathcal{B}_2$ with maximum degree Δ such that $mS_k(G) \geq \lceil \frac{3\Delta}{2} \rceil$.*

Proof. To obtain the graph G, let us start with the cycle (u_2, u_1, v_1, v_2) and, for $3 \leq i \leq \Delta$, add the edges u_iv_i, u_1u_i and v_1v_i (where u_i and v_i are new vertices). It is easy to see that G belongs to $\mathcal{B}_2$. By Claim 2.1, for every $2 \leq i \leq \Delta$, we have $\ell(u_1u_i) \neq \ell(v_1v_i)$ for any k-labeling ℓ. Therefore, in any labeling, u_1 or v_1 must be incident to at least $\lfloor \frac{\Delta-1}{2} \rfloor$ edges not labeled with 1 (otherwise, we would have $\ell(u_1u_i) = \ell(v_1v_i)$ for some $2 \leq i \leq \Delta$). This implies that the following 2-labeling is optimal. Each edge u_iv_i for every $1 \leq i \leq \Delta$ is labeled with 1 by ℓ. Then, for $2 \leq i \leq \lceil (\Delta - 1)/2 \rceil + 1$, $\ell(u_1u_i) = 1$ and $\ell(v_1v_i) = 2$, and for every $\lceil (\Delta - 1)/2 \rceil + 2 \leq i \leq \Delta$, we have $\ell(u_1v_i) = 2$ and $\ell(v_1v_i) = 1$. If $\Delta - 1$ is odd, then $mS(G, \ell) = c_\ell(v_1) = 2\lceil (\Delta - 1)/2 \rceil + \lfloor (\Delta - 1)/2 \rfloor + 1 = 3\lceil (\Delta - 1)/2 \rceil = \lceil 3\Delta/2 \rceil$ (while $c_\ell(u_1) = c_\ell(v_1) - 1$). If $\Delta - 1$ is even, then, to avoid $c_\ell(u_1) = c_\ell(v_1)$, let us relabel u_1u_2 and v_1v_2 so that $\ell(u_1u_2) = 2$ and $\ell(v_1v_2) = 1$. In this case, $mS(G, \ell) = c_\ell(u_1) = 2((\Delta - 1)/2 + 1) + (\Delta - 1)/2 = 3\lfloor (\Delta - 1)/2 \rfloor + 2 = \lceil 3\Delta/2 \rceil$. □

4.1.2. Graphs of $\mathcal{B}_3$

In [10], the authors proved that the graphs of $\mathcal{B}_3$, i.e., the bipartite graphs G with $\chi_\Sigma(G) = 3$, are precisely the so-called *odd multi-cacti.*

In what follows, let us give a constructive definition of odd multi-cacti together with the notations that will be used further in this section. The odd multi-cacti are exactly the graphs that can be obtained by the following recursive construction (see Figure 2 for an illustration), which actually produces graphs whose edges are colored either *red* or *green.* This (not necessarily proper) edge-coloring is then used by the process itself.

- Any cycle of length $n \equiv 2 \mod 4$, $n \geq 6$, together with a proper red-green coloring of its edges (this is actually the only point of the construction where the edge-coloring is proper) is an odd multi-cactus. Let C_0 be the *root cycle* of the odd multi-cactus. Let also uv be one green edge of C_0 that is called the *starting edge.*

- Given an odd multi-cactus G', with its root cycle C_0, its starting edge uv and its red-green (not necessarily proper) edge-coloring, it can be extended to a larger odd multi-cactus G as follows. Let xy be any green edge of G' with the extra constraint that if $xy \in E(C_0)$, then $xy = uv$. Then, G is obtained by *attaching* a new path P with length at least 5 congruent to 1 modulo 4 to xy. By "attaching", we mean identifying x and an end-vertex of P, and identifying y and the second end-vertex of P. Moreover, the edges of the path P are alternatively colored red and green, in such a way that the two end edges (i.e., those incident to x or y) are colored red. Let $C(P)$ be the cycle of G induced by the edges of P together with xy and let xy be

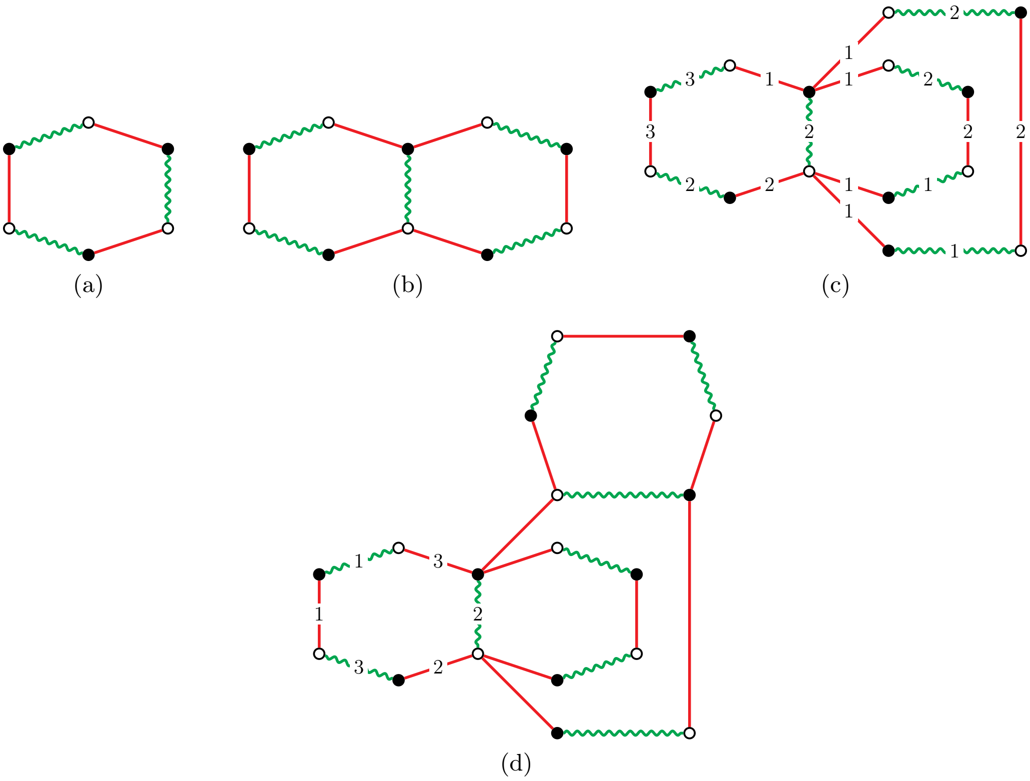


Figure 2: Constructing an odd multi-cactus through several steps, from the red-green C_6 (a). Red-green paths with length at least 5 congruent to 1 modulo 4 are being repeatedly attached onto green edges through steps (b) to (d). Figure (a) gives an example of the labeling of C_0 following the pattern $23113(2211)^z2$ (note that this is the labeling before labeling the next paths, i.e., before switching the label of edge uv to 1). Figure (c) gives an example of the labeling of C_0 in the particular case when C_0 has length 6 and $|\mathcal{P}_1| = 2$. Wiggly edges are green edges.

called the *parent edge* of P (and of $C(P)$). Moreover, if $xy = uv$, then let C_0 be the *parent cycle* of $C(P)$. Otherwise (if $xy \notin E(C_0)$), let P' be the first attached path containing xy during the construction of G'; then let $C(P')$ be the *parent cycle* of $C(P)$. Finally, the cycle C_0 remains the root cycle of G and uv remains its starting edge.

First, let us point out a trivial but important property of odd multi-cacti.

Claim 4.3. *Let G be any odd multi-cactus with its red-green edge-coloring. For any green edge $xy \in E(G)$, we have $d(x) = d(y)$.*

The recursive definition of odd multi-cacti provides the following natural tree-like structure of their induced cycles. Let G be any odd multi-cactus with root cycle C_0. The “parent cycle” relation allows to define the tree $T(G)$ whose nodes are the induced cycles of G, rooted in C_0, and such that $CC' \in E(T(G))$ if C is the parent cycle of C' or *vice versa* (abusing the notations, let us identify the induced cycles of G (and corresponding attached paths) and the nodes of $T(G)$). We make use of the tree $T(G)$ only to formally define a suitable ordered partition of the paths recursively attached to obtained G from C_0. Precisely, for any non-leaf node C of $T(G)$, let $\mathcal{P}_C$ be the set of paths P such that the induced cycle C is the parent cycle of $C(P)$ in G (note that, if $G \neq C_0$, then $\mathcal{P}_C \neq \emptyset$ if and only if C is not a leaf-node of $T(G)$ or $C = C_0$). A *valid path-partition* of G is then defined as $\mathcal{P}(G) = (\mathcal{P}_{C_0}, \mathcal{P}_{C_1}, \ldots, \mathcal{P}_{C_f})$, where $(C_0, \ldots, C_f)$ is any BFS ordering of the non-leaf nodes of $T(G)$ rooted in C_0.

Our main result in this section is that, for an odd multi-cactus G, the parameter $mS_3(G)$ is always one of two possible values.

Theorem 4.4. *For every odd multi-cactus G with maximum degree $\Delta \geq 3$, we have $\Delta+1 \leq mS_3(G) \leq \Delta + 2$.*

Proof. The lower bound follows from the fact that, by construction, for every green edge uv we have $d(u) = d(v)$. Moreover, the maximum degree is attained at some green edge, say xy, i.e., $d(x) = d(y) = \Delta$. Thus, in any labeling ℓ of G, assuming $c_\ell(x) < c_\ell(y)$, we must have $\Delta \leq c_\ell(x)$, which implies that $\Delta + 1 \leq c_\ell(y)$.

We now focus on proving the upper bound, i.e., on proving that G admits a 3-labeling ℓ where the maximum color $c_\ell(v)$ over all vertices v is at most $\Delta + 2$.

Let G be any odd multi-cactus with maximum degree $\Delta \geq 3$ (in particular, $T(G)$ is not reduced to a single vertex) with its red-green edge-coloring. Let C_0 be its root cycle, let uv be its starting edge, and let $(\mathcal{P}_1, \ldots, \mathcal{P}_q)$ be some valid path-partition of G.

We deduce a 3-labeling ℓ with maximum color $\Delta + 2$ of G by starting from one of C_0, then extending it to the edges of the paths in $\mathcal{P}_1$, then to the edges of the paths in $\mathcal{P}_2$, and so on. So we must show that, at every step, an extension does exist, and that, in particular, there is one for which the maximum color does not exceed $\Delta + 2$.

Let y be the size of C_0. Let us first label the edges of C_0 as follows. Let $(e_1 = vu, e_2 = uv_1, e_3 = v_1v_2, \ldots, e_y = v_{y-2}v)$ be the edges of C_0 in order with the starting edge as first edge (i.e., e_2 is incident to u, e_y is incident to v, and $y \equiv 2 \mod 4$ and $y \geq 6$). We label them by applying the pattern $23113(2211)^z2$ (where x^s denotes the concatenation of s copies of the string x, and $z = (y-6)/4$; see Figure 2 (d) for an example). That is, $\ell(e_1) = 2, \ell(e_2) = 3, \ell(e_3) = 1, \ell(e_4) = 1, \ell(e_5) = 3, \ldots, \ell(e_y) = 2$. Note that, so far, $c_\ell(u) = 5, c_\ell(v) = 4, c_\ell(v_1) = 4, c_\ell(v_{y-2}) = 3$ (or $c_\ell(v_{y-2}) = 5$ if $y = 6$) and $c_\ell(x) \leq 5$ for all $x \in V(C_0)$ (which is at most $\Delta + 2$ since $\Delta \geq 3$). We now sequentially extend this labeling to all edges of G in such a way that the labels of the edges in $E(C_0) \setminus \{e_1\}$ are never modified (except in one particular case if $|E(C_0)| = 6$ and $|\mathcal{P}_1| = 2$). Also, the only other edges that will be labeled 3 are green edges whose two ends have both degree 2. This way, the only possible red edge with label 3 will be one of the root cycle.

The second phase of the labeling consists in extending our current labeling ℓ to all paths in $\mathcal{P}_1$ (i.e., all paths that have been attached to the starting edge uv). First, switch the label of uv from 2 to 1. Then, for every path $P \in \mathcal{P}_1$ (recall that P has length l_P at least 5 and congruent to 1 modulo 4 and has been attached to the starting edge uv), let us label its edges in order from u to v by applying the pattern $(1122)^{z_P}1$ (where $z_P = (l_P - 1)/4$). Note that the resulting labeling ℓ ensures that, for any $P \in \mathcal{P}_1$ and any internal vertex x of P, the color of x is at most 4 and distinct from the color of its neighbors. Similarly, u and v have distinct colors ($c_\ell(u) = 4 + |\mathcal{P}_1|$ and $c_\ell(v) = 3 + |\mathcal{P}_1|$) that are distinct from the colors of their respective neighbors but in one particular case: when $|\mathcal{P}_1| = 2$ and $y = 6$, in which case $c_\ell(v) = 5 = c_\ell(v_{y-2})$. Except in the latter case, the resulting labeling ℓ is proper and the maximum color is $c_\ell(u) = 4 + |\mathcal{P}_1| \leq \Delta + 2$. In the pathological case, i.e., when $|\mathcal{P}_1| = 2$ and $y = 6$, we relabel the edges $(e_1, \ldots, e_6)$ of C_0 by applying the pattern 213322 in order (see Figure 2c for an example). As a result we get $c_\ell(u) = 5, c_\ell(v) = 6, c_\ell(v_1) = 4$ and $c_\ell(v_{y-2}) = 4$, and ℓ is thus proper. Furthermore, the maximum color is $6 \leq \Delta + 2$ since $\Delta \geq 4$ (because $|\mathcal{P}_1| = 2$).

Recall that we are given a path-partition $(\mathcal{P}_1, \dots, \mathcal{P}_q)$ of G. For every $1 \le i \le q$, let G_i be the subgraph of G induced by the vertices of C_0 and of all paths in $\mathcal{P}_j$ for every $1 \le j \le i$. A *pending path* of G_i is any path P of $\bigcup_{1\le j\le i} \mathcal{P}_j$ such that $C(P)$ has exactly two vertices of degree more than 2 in G_i. Note that, for any pending path P of G_i, either $C(P)$ is a leaf of $T(G)$ or there exists $i < j \le q$ such that $\mathcal{P}_j$ is exactly the set of paths attached to some green edges (different from its parent edge) of $C(P)$.

If $q = 1$, then the labeling obtained so far satisfies the statement for $G = G_1$ and we are done. Otherwise, let $1 \le i < q$ and assume by induction on i that we have a labeling ℓ of G_i such that:

- ℓ is a 3-labeling where the only red edges assigned label 3 belong to the root cycle; any green edge labeled with 3 has both its ends of degree 2; ℓ is proper and there is thus no conflict; $c_\ell(v) \le \Delta + 2$ for every $v \in V(G_i)$; and
- for every pending path P of G_i with parent edge xy, the edges of P are labeled (from y to x or from x to y) following the pattern $(1122)^{z_P}1$ or $(2211)^{z_P}2$ (where P has length $4z_P + 1 \ge 5$), and $|c_\ell(x) - c_\ell(y)| = 1$. In particular, every two "consecutive" red edges of P are assigned distinct labels in $\{1, 2\}$.

These properties are clearly satisfied by the labeling ℓ defined for G_1. Let us assume that these properties hold for a labeling ℓ obtained for G_{i-1} (for some $1 < i \le q$) and let us show how to extend it to a labeling (with these properties) of G_i (such a labeling for $G_q = G$ will clearly satisfy the statement of the theorem).

By definition, the paths of $\mathcal{P}_i$ are the ones that correspond to the induced cycles with a common parent cycle $C \ne C_0$, i.e., let C be the induced cycle of G_{i-1} such that $\mathcal{P}_i = \mathcal{P}_C$. Hence, the paths of $\mathcal{P}_i$ have been attached to some green edges (different from the parent edge) of some pending path P of G_{i-1} (such that $C(P) = C$). Let $P = (y, x_1, y_1, x_2, y_2, \dots, x_k, y_k, x)$ (where P has length $2k+1 \ge 5$). That is, $\{x_j y_j \mid 1 \le j \le k\}$ is the set of green edges of P different from $x_0y_0 = xy$, where both yx_1 and y_kx are joined by a red edge, and each y_jx_{j+1} is also a red edge (for $j \in \{1, \dots, k-1\}$). For each green edge x_jy_j ($j \in \{1, \dots, k\}$) of P, let us denote by $n_j \ge 0$ the number of paths of $\mathcal{P}_i$ that are attached to x_jy_j. Note that $d_G(x_j) = d_G(y_j) = 2 + n_j$ for every $1 \le j \le k$ (where $d_G(b)$ denotes the degree of b in G).

Let us consider the green edges $x_1y_1, x_2y_2, \dots, x_{k-1}y_{k-1}$ in order for $j = 1$ to $k-1$. For each x_jy_j of them, we label the edges of the attached paths following one of the following two extension schemes:

- *Scheme A*: We set the label of x_jy_j to 1. Then, we label the edges of all paths attached to x_jy_j following the pattern $(1122)^{z_P}1$ from x_j to y_j or *vice versa*, so that no inner vertex is involved in a conflict.
- *Scheme B*: We apply Scheme A, but change the label of x_jy_j to 2.

Note that Schemes A and B can be applied whatever the degree of x_j and y_j is (i.e., whatever n_j is). It is easy to see that for each of the two extension schemes, labeling the edges of the paths either as going from x_j to y_j or *vice versa* indeed raises no conflict, unless x_j and y_j have the same color, a contradiction. Whenever applying one of the two schemes in what follows, it is thus understood that this is done so that no conflict involving an inner vertex of a path occurs. Also, it is important to note that every extension scheme alters the colors of x_j and y_j the same way.

The extension of ℓ to the successive x_jy_j ($j = 1, \dots, k-1$) goes as follows. Assume we are currently dealing with x_jy_j. Let us first apply extension Scheme A. At most two conflicts may occur: along the edge $y_{j-1}x_j$ and/or along the edge y_jx_{j+1}. We first deal only with the conflict between y_{j-1} and x_j. If $c_\ell(y_{j-1}) = c_\ell(x_j)$, then let us apply extension Scheme B instead. It is easy to check that the resulting partial labeling satisfies all desired properties (in particular, the maximum color is at most $\Delta + 2$) but, possibly, there is a conflict between y_j and x_{j+1}; this conflict will be dealt with later, when dealing with the next green edge $x_{j+1}y_{j+1}$.

We are thus left with extending the labeling to the n_k paths attached to x_ky_k. Here, we might need to use a third extension scheme for a green edge x_jy_j:

- *Scheme C*: We set the label of x_jy_j to 2. Then, we label the edges of all paths attached to x_jy_j following the pattern $(1122)^{z_P}1$ but an arbitrary one P^* of them that is labeled following the pattern $(2211)^{z_P}2$ (all paths being labeled either from x_j to y_j or *vice versa*, so that no inner vertex is involved in a conflict).

Note that Scheme C requires $n_j > 0$ to be different from Scheme B. When $n_j = 0$, i.e., x_j, y_j have degree 2, we consider the following scheme instead:

- *Scheme C'*: We set the label of x_jy_j to 3.

An important point to raise is that, when applying Scheme C' under all conditions maintained so far (in particular, without loss of generality the edge incident to x_j different from x_jy_j is labeled 1 while the edge incident to y_j different from x_jy_j is labeled 2), the colors of x_j and y_j do not exceed 5, which is at most $\Delta + 2$ since $\Delta \geq 3$.

Now consider x_ky_k. If $d_G(x_k) = d_G(y_k) < \Delta$, then we are done, because by applying one of Schemes A, B, C (or C') above, we can extend the labeling to all paths without creating conflicts, and with having at most three edges labeled 2 incident to one of x_k and y_k, implying that their colors are at most $\Delta+2$ and differ by 1 (unless the degree of x_k and y_k is 2, in which case Scheme C' might have introduced a 3, a situation we have discussed above).

So let us assume that $d_G(x_k) = d_G(y_k) = \Delta$. We can actually assume that there is no $j \in \{1, \dots, k\}$ such that $d_G(x_j) = d_G(y_j) < \Delta$, as otherwise we could first extend the labeling to the paths attached to $x_1y_1, x_2y_2, \dots, x_{j-1}y_{j-1}$ following this order, then to those attached to $x_ky_k, x_{k-1}y_{k-1}, \dots, x_{j+1}y_{j+1}$ following this order, and then to those attached x_jy_j via one of Schemes A, B, C (or C'), resulting in x_j and y_j verifying the same conditions as above.

So let us assume that $d_G(x_j) = d_G(y_j) = \Delta$ for every $j \in \{1, \dots, k\}$, and assume that only the labeling of the n_k paths attached to x_ky_k remain to be corrected. Note that, so far, every red edge of P has kept the same label as in G_{i-1}. This implies that, for every j, the red edge incident to x_j on P must be labeled differently than the red edge incident to y_j on P. Moreover, the way we have modified the labeling of G_{i-1} so far also ensures that the colors of x_j and y_j differ by 1.

Let us assume that the red edge incident to y_k on P (that is the edge y_kx_0) is labeled 2, while the red edge incident to x_k on P (that is the edge $y_{k-1}x_k$) is labeled 1. Propagating this assumption along P from $x = x_0$ to $y = y_0$, the labels of the red edges of P must alternate, and, due to its length, y_0x_1 is labeled 2 as well. As a first extension attempt, let us apply Scheme A to x_ky_k. The color of x_k then becomes Δ while the color of y_k becomes $\Delta + 1$. If no conflict arises, then we are done. Otherwise, there are two possible sources for conflict:

- The color of x_0 is also $\Delta + 1$. Then we apply Scheme B at $x_k y_k$ instead. Now x_k has color $\Delta + 1$ while y_k has color $\Delta + 2$. If no further conflict arises, then we are done. Otherwise, then it must be because y_{k-1} has color $\Delta + 1$. Because $y_{k-1}x_k$ is labeled 1, it means that $x_{k-1}y_{k-2}$ is labeled 2, and due to how we have been extending the labeling, we deduce that x_{k-1} has color $\Delta + 2$.

  Now come back to the moment where we have extended the labeling to the n_{k-1} paths attached to $x_{k-1}y_{k-1}$. By the color assumptions we have, we deduce that we have here applied Scheme B. If applying Scheme A instead does not raise a conflict, then we are done because the color of y_{k-1} would become Δ (while that of x_{k-1} would become $\Delta + 1$), a favorable case for extending the labeling to the n_k paths attached to $x_k y_k$. Otherwise, then it must be because y_{k-2} has color $\Delta + 1$. Again, since $x_{k-1}y_{k-2}$ is labeled 2, then $x_{k-2}y_{k-3}$ must be labeled 1, and x_{k-2} must have color Δ, since the colors of x_{k-2} and y_{k-2} differ by 1.

  Going on considering green edges like this as going along P from $x_k y_k$ to $x_1 y_1$, we either find a green edge for which a different pair $(c_\ell(x_k), c_\ell(y_k))$ of colors at most $\Delta + 2$ can be reached (by employing a different extension scheme), which would lead to a favorable case for extending the labeling to all attached paths, or we successively deduce that $(c_\ell(x_{k-1}), c_\ell(y_{k-1})) = (\Delta + 2, \Delta + 1)$, $(c_\ell(x_{k-2}), c_\ell(y_{k-2})) = (\Delta, \Delta + 1)$, $(c_\ell(x_{k-3}), c_\ell(y_{k-3})) = (\Delta + 2, \Delta + 1)$, $(c_\ell(x_{k-4}), c_\ell(y_{k-4})) = (\Delta, \Delta + 1)$, and so on. Due to the length of P, we eventually deduce that $(c_\ell(x_1), c_\ell(y_1)) = (\Delta + 2, \Delta + 1)$, and that y_0 has color $\Delta + 1$. Then both x_0 and y_0 have color $\Delta + 1$, which is a contradiction.

- The color of y_{k-1} is also Δ. Then we apply Scheme B to $x_k y_k$ instead. Now x_k has color $\Delta + 1$ while y_k has color $\Delta + 2$. Now, if another conflict arises, then it is because x_0 has color $\Delta + 2$. By arguments as in the previous case, we can get a favorable case by altering the colors of a previous green edge (employing extension Scheme A instead of Scheme B, and *vice versa*), unless $(c_\ell(x_{k-1}), c_\ell(y_{k-1})) = (\Delta + 1, \Delta)$, $(c_\ell(x_{k-2}), c_\ell(y_{k-2})) = (\Delta + 2, \Delta + 1)$, $(c_\ell(x_{k-3}), c_\ell(y_{k-3})) = (\Delta + 1, \Delta)$, $(c_\ell(x_{k-4}), c_\ell(y_{k-4})) = (\Delta + 2, \Delta + 1)$, and so on. Due to the length of P, we eventually deduce that $(c_\ell(x_1), c_\ell(y_1)) = (\Delta + 1, \Delta)$, and that y_0 has color $\Delta + 2$. Then both x_0 and y_0 have color $\Delta + 2$, a contradiction.

The same type of arguments also apply when the first edge and the last edge of P are labeled 1. In all cases, we can extend the labeling to all paths attached to P, in such a way that no color exceeds $\Delta + 2$. This concludes the proof. □

In the line of Theorem 4.4, the next natural step would be to investigate, given an odd multi-cactus G, whether determining $mS_3(G)$ is hard or not. A consequence of Theorem 3.3 is that this value can be determined in polynomial time.

Corollary 4.5. *The problem of deciding $mS_k(G)$ can be solved in polynomial time in the class of odd multi-cacti G (i.e., when $G \in \mathcal{B}_3$).*

Proof. This comes from the fact that $tw(G) = 2$ for any odd multi-cactus $G \in \mathcal{B}_3$. Indeed, G is nothing but a collection of induced cycles every two of which share at most one edge. The fact that such a graph has treewidth 2 can easily be proved by induction on the number of induced cycles. Theorem 3.3 then applies to G, proving the claim. □

4.2. Trees

In this section, we focus on trees T (recall that $\chi_\Sigma(T) \leq 2$ for any nice tree T [3]). The main result of this section is that $mS_2(T)$ is always one of three possible values, each of which can be reached.

Theorem 4.6. *For any $k \geq 2$ and any nice tree T with maximum degree Δ, then $mS_k(T) \in \{\Delta, \Delta+1, \Delta+2\}$. Moreover, all these values are reached.*

Proof. The lower bound Δ holds by Claim 2.2. To prove the upper bound, let us design a labeling process that will achieve a 2-labeling ℓ with $mS(T, \ell) \leq \Delta + 2$. Let us root T in any arbitrary node r. The process will consider all vertices one by one in a BFS ordering (i.e., a vertex at distance d from the root is considered once all vertices at distance less than d from the root have been considered). Moreover, once a vertex v has been considered, all its incident edges are labeled (and their labels will never be modified anymore), it is not in conflict with its parent (if $v \neq r$), and its color $c_\ell(v)$ is at most $\Delta + 2$.

Start by labeling all edges incident to r with 1 (so $c_\ell(r) = d(r) \leq \Delta$). Now, let $v \neq r$ be any vertex such that its parent u has already be considered. Hence, all edges incident to u have received a label (in $\{1, 2\}$), and so, $c_\ell(u)$ is well defined. Let d be the number of children of v.

- If $d > 0$, then there are two cases to be considered. If $d + \ell(uv) = c_\ell(u)$ then label all edges between u and its children with 1 but one such edge that is labeled 2 (i.e., $c_\ell(v) = d + \ell(uv) + 1$). Otherwise, label all edges between u and its children with 1 (i.e., $c_\ell(v) = d + \ell(uv)$). In both cases, $c_\ell(v) \neq c_\ell(u)$. Moreover, v is incident to at most two edges labeled 2 (possibly its parent edge and one other incident edge) and so $c_\ell(v) \leq d(v) + 2 \leq \Delta + 2$.
- If $d = 0$ (i.e., v is a leaf with color $\ell(uv)$), then note that u has degree at least 2 since T is nice and so $c_\ell(v) \neq c_\ell(u)$.

We conclude with the last part of the statement. Let $\Delta > 1$. It is easy to see that any star S_Δ with maximum degree Δ is locally irregular, and thus $mS_k(S_\Delta) = \Delta$ for every $k \geq 1$. The fact that there are trees T with maximum degree Δ such that $mS_k(T) = \Delta+1$ or $mS_k(T) = \Delta + 2$ follows from Corollary 4.8 and Proposition 4.9 below. $\square$

Following Theorem 4.6, we say that a tree with maximum degree Δ is of *type* x for $x \in \{\Delta, \Delta+1, \Delta+2\}$ if $mS_2(T) = x$. The next natural step in the line of Theorem 4.6 would be to provide a full characterization of the trees of type Δ, $\Delta+1$ or $\Delta+2$. A consequence of the polynomial-time algorithm given in Section 3.2 is that an algorithmic characterization exists. But we wonder whether a more natural characterization exists, such as a characterization in terms of particular subtrees. Towards such a characterization, we provide, in the rest of this section, sufficient conditions for a tree to be of type Δ, $\Delta+1$ or $\Delta+2$.

We start off by providing an easy condition in which a tree cannot be of type Δ.

Observation 4.7. *Let $\Delta \geq 2$ and T be any tree with maximum degree Δ having two adjacent vertices of degree Δ. Then, $mS_k(T) \geq \Delta+1$ for any $k \geq 2$.*

Proof. The two adjacent vertices with degree Δ must have different colors whatever be the labeling. Hence, at least one of them must have an incident edge labeled with at least 2. Hence, its color is at least $\Delta + 1$. $\square$

Corollary 4.8. *For every $\Delta \geq 2$, there are trees with maximum degree Δ of type $\Delta + 1$.*

Proof. Consider a bistar with two adjacent vertices of degree Δ, i.e., a tree obtained from two adjacent vertices by making each of them adjacent to $\Delta - 1$ leaves. $\square$

In what follows, we introduce a family of trees of type $\Delta + 2$. Let us introduce some notations. Let $\Delta \geq 3$. Let F_Δ be the rooted tree such that its root has degree $\Delta - 1$ and each neighbor of the root has Δ neighbors each of which (except the root) is a leaf. Let H_Δ be the tree obtained from two copies of F_Δ by making their roots adjacent.

Proposition 4.9. *For every $\Delta \geq 3$, every tree T, with maximum degree Δ, containing H_Δ as a subtree is of type $\Delta + 2$.*

Proof. Let u and v be the roots of the two copies of F_Δ. Note that they have degree Δ in T. In any k-labeling, since they must have different colors, at least one of them, say u, must have at least one incident edge (not uv) labeled 2 (if one edge is labeled with more than 2, then the maximum color is already at least $\Delta + 2$). Let us assume that u has exactly one incident edge labeled with 2 and all others are labeled with 1 (since otherwise, the color of u would already be at least $\Delta + 2$) so that the color of u is $\Delta + 1$. Let ux be the edge labeled with 2. Now, x has degree Δ, it has at least one incident edge labeled 2 (so its color is at least $\Delta + 1$) and cannot have color $\Delta + 1$. Hence its color is at least $\Delta + 2$ and $mS_k(T) \geq \Delta + 2$. The equality comes from Theorem 4.6. $\square$

4.3. Using larger labels in trees

In the previous section, we have studied 2-labelings of nice trees T, showing in Theorem 4.6 that $mS_2(T)$ is essentially one of three possible values (function of the maximum degree). A natural question to ask is whether the use of larger labels can lead to a decrease of the maximum color. This question makes more particularly sense for the trees of type $\Delta + 2$, since their value of mS_2 is the worst one for a tree with maximum degree Δ.

The next result shows several things. First, that, for trees, using larger labels can indeed allow to decrease the maximum color. Second, and more importantly, that there are graphs (and even trees) for which, in order to make the maximum color decrease, we have to employ arbitrarily large labels.

Theorem 4.10. *For every $k \geq 2$, there exists a tree T_k such that $mS_{k+1}(T_k) = mS_k(T_k) - 1$.*

Proof. For $k = 2$, the tree T_2 depicted in Figure 3 satisfies $mS_2(T_2) = 6$ and $mS_3(T_2) = 5$. Indeed, in any 2-labeling ℓ, by Claim 2.4, one of w_1x_1 or w_2x_2 must be labeled with 2, w.l.o.g., say $\ell(w_1x_1) = 2$. Then, one of $v_1'w_1$ or v_1w_1, say v_1w_1, must be labeled with 1 (as otherwise w_1 would have color 6). If $c_\ell(w_1) = 4$, then v_1 cannot have color 4 and one edge incident to v_1 (but w_1v_1), say u_1v_1, must be labeled 2. Now v_1 has color 5, and u_1 has color at least 5, so $mS(T_2, \ell) \geq 6$. If $c_\ell(w_1) = 5$, then w.l.o.g., $\ell(v_1', w_1) = 2$, and v_1' has color at least 5. Again $mS(T_2, \ell) \geq 6$. It is easy to see that $mS_2(T_2) \leq 6$. Moreover, labeling the edges w_1x_1, x_1x_2, x_2w_2 with $3, 1, 1$ in order can easily been extended to a 3-labeling ℓ' (using label 3 only once) with $mS(T_2, \ell') = 5$, hence $mS_3(T_2) \leq 5$ (the equality holds since there are two adjacent vertices with degree 4).

For any $k \geq 3$, let us build a tree T_k with maximum degree $\Delta > 2k$ such that $mS_{k+1}(T_k) = \Delta$ and $mS_k(T_k) = \Delta + 1$. We use some gadgets similar with (but more general than) the ones used in Theorem 3.2.

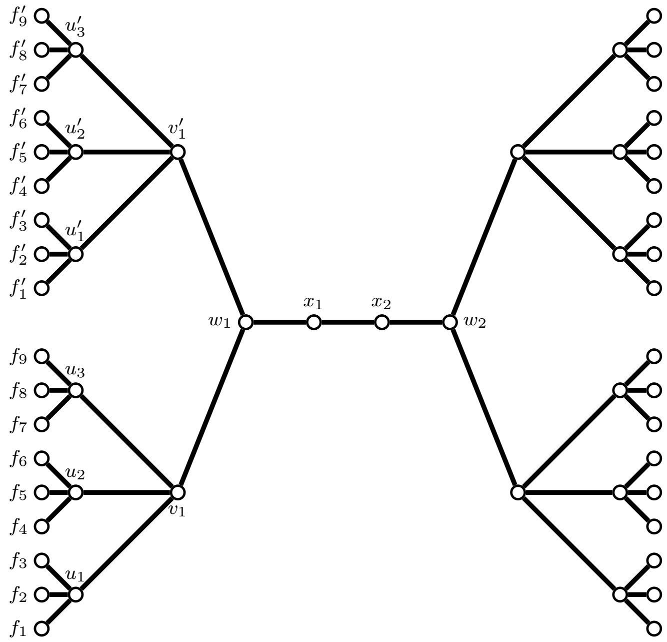


Figure 3: The tree T_2 with $mS_2(T_2) = 6$ and $mS_3(T_2) = 5$.

Let us first define the gadget $H(d)$ (that depends on k and $\Delta > 2k$ that are fixed) for $d \in \{\Delta, \Delta - 1, \ldots, \Delta - k\}$. $H(\Delta)$ is a star with Δ leaves, rooted at one of the leaves, denoted as r and with center of degree Δ. For every $\Delta > d \geq \Delta - k$, let us define $H(d)$ as follows. Start with one copy of $H(d')$ for every $d' \in \{\Delta, \Delta - 1, \ldots, d+1\}$; then identify all their roots, denoting the obtained vertex c as the center of $H(d)$ (which is then adjacent to all centers of the $H(d')$'s). Finally, let us add another $d - (\Delta - d) = 2d - \Delta > 0$ (because $\Delta > 2k$ and $d \geq \Delta - k$) leaves adjacent to c to make sure that c has degree d. The root r of $H(d)$ is any leaf adjacent to c.

Claim 4.11. *Let $d \in \{\Delta, \Delta - 1, \ldots, \Delta - k\}$. Then, for any k-labeling ℓ of $H(d)$, we have $mS(H(d), \ell) = \Delta$ if and only if $\ell(e) = 1$ for any $e \in E(H(d))$. Particularly, the label of the edge between the root r and the center c of $H(d)$ satisfies $\ell(rc) = 1$ and the color of the center c is d.*

Proof of the claim. It is obviously true for $d = \Delta$. By induction on d, let us assume that the statement holds for every d' with $d < d' \leq \Delta$.

If each edge of $H(d)$ is labeled with 1, then, for every $d' \in \{\Delta, \Delta - 1, \ldots, d+1\}$, the maximum color of the vertices in the $H(d')$'s contained in $H(d)$ is at most Δ. Moreover, their centers are colored d' for $d' \in \{\Delta, \Delta - 1, \ldots, d+1\}$ and, by the induction hypothesis, there are no conflicts in the $H(d')$'s. Note that the center c of $H(d)$ has degree d. So it is colored d and no conflict occurs with its neighbors.

If $mS(H(d), \ell) = \Delta$ for some k-labeling ℓ, then the maximum color of every $H(d')$ for each $d' \in \{\Delta, \Delta - 1, \ldots, d+1\}$ contained in $H(d)$ is also Δ. By the induction hypothesis, for each $d' \in \{\Delta, \Delta - 1, \ldots, d+1\}$, every edge in the copy of $H(d')$ is labeled with 1. Moreover, for every $d' \in \{\Delta, \Delta - 1, \ldots, d+1\}$, the center of $H(d')$ is colored with d'. So the center c of $H(d)$ (which has degree d) has to be colored d, as otherwise it would get a color more than Δ to avoid conflicts with its neighbors. Hence, the only way that $mS(H(d), \ell) = \Delta$ is that all the edges incident to c are also labeled 1. $\diamond$

Now, let D_k be the tree built as follows (note that it also depends implicitely on Δ).

D_k is obtained from one copy of $H(d)$ for every $d \in \{\Delta - k, \Delta - k + 1, \ldots, \Delta - 2\}$ and from $\Delta - 2k$ extra copies of $H(\Delta - 2)$ by identifying all their roots into one single vertex c, called the center of D_k. Finally, add one leaf adjacent to c (so that c has now degree $\Delta - k$), this leaf being the root of D_k.

Claim 4.12. *Let D_k be rooted at r and centered at c. Then $mS_k(D_k) = mS_{k+1}(D_k) = \Delta$. Moreover, the unique k-labeling ℓ with $mS(D_k, \ell) = \Delta$ is such that $\ell(rc) = k$ and $\ell(e) = 1$ for every edge e in $E(D_k) \setminus \{rc\}$, and any $(k+1)$-labeling ℓ with $mS(D_k, \ell) = \Delta$ is such that $\ell(rc) \in \{k, k+1\}$ and $\ell(e) = 1$ for every edge e in $E(D_k) \setminus \{rc\}$*

Proof of the claim. Since any labeling ℓ of D_k such that $mS(D_k, \ell) = \Delta$ induces a labeling with maximum color Δ for each of the copies of $H(d)$ ($d \in \{\Delta - k, \Delta - k + 1, \ldots, \Delta - 3, \Delta - 2\}$), by the previous claim all edges $e \in E(D_k) \setminus \{rc\}$ must be labeled with 1. Moreover, the center of a copy of $H(d)$ for $d \in \{\Delta - k + 1, \Delta - k + 2, \ldots, \Delta - 3, \Delta - 2\}$ must have color d. Since the center c of D_k is adjacent to the centers of the copies of the $H(d)$'s and c has degree $\Delta - k$ and is adjacent to $\Delta - (k + 1)$ edges labeled with 1 (the edges incident to the centers of the copies of the $H(d)$'s), the last edge rc can only be labeled with k or $k + 1$ to ensure that $c_\ell(c) \leq \Delta$ and $c_\ell(c)$ is different from any value in $\{\Delta - k, \Delta - k + 1, \ldots, \Delta - 3, \Delta - 2\}$. ◇

Now we are ready to define the tree T_k and prove that $mS_{k+1}(T_k) = \Delta < mS_k(T_k) = \Delta + 1$. Let T_k be obtained from two copies of D_k by adding one edge incident to both roots of the copies of D_k. Let c_i, r_i, $i \in \{1, 2\}$ be respectively the center and the root of the two copies of D_k (so T_k is obtained by adding the edge $r_1 r_2$). By the previous claim, any labeling ℓ of T_k such that $mS(T_k, \ell) = \Delta$ must be such that $\ell(r_i c_i) \in \{k, k + 1\}$ for each $i \in \{1, 2\}$. If ℓ is a k-labeling, then no edge can be labeled with $k + 1$ and we must have $\ell(r_1 c_1) = \ell(r_2 c_2) = k$ but this would imply that $c_\ell(r_1) = c_\ell(r_2)$. Hence, $mS_k(T_k) > \Delta$ (and it is easy to see that $mS_k(T_k) \leq \Delta + 1$). On the other hand, if ℓ is a $(k+1)$-labeling, then setting $\ell(r_1 c_1) = k + 1$ and $\ell(r_2 c_2) = k$ leads to a labeling of T_k with maximum color Δ. □

4.4. Using larger labels in general graphs

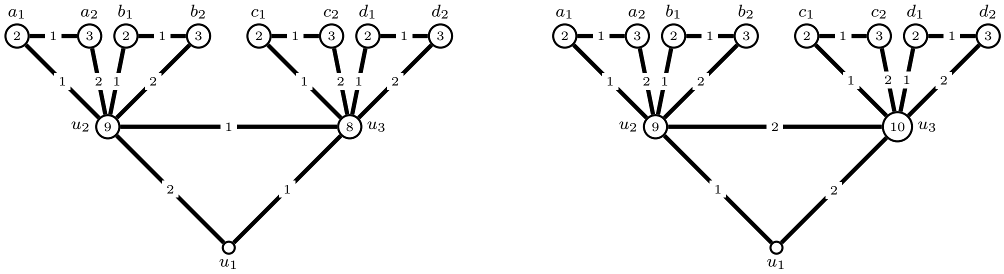
In this section, we construct graphs G verifying $mS_2(G) = 2\Delta(G)$ and $mS_3(G) = \Delta(G)$, see final Theorem 4.18. We obtain these graphs by connecting several smaller graphs in some fashion. These smaller graphs are depicted in figures all along this section. Whenever dealing with their vertices and edges later, we implicitly do so using the terminology used in the corresponding figure. Most of our graphs will contain *inputs* and *outputs*, which are pending edges which will serve for the connections.

Let T_2 be the graph with 11 vertices and 15 edges depicted in Figure 4, obtained from five edge-disjoint triangles (u_1, u_2, u_3), (u_2, a_1, a_2), (u_2, b_1, b_2), (u_3, c_1, c_2) and (u_3, d_1, d_2). Let u_1 be the *root vertex* of T_2. It has the following labeling properties.

Lemma 4.13. *In every 2-labeling ℓ of T_2, we have:*

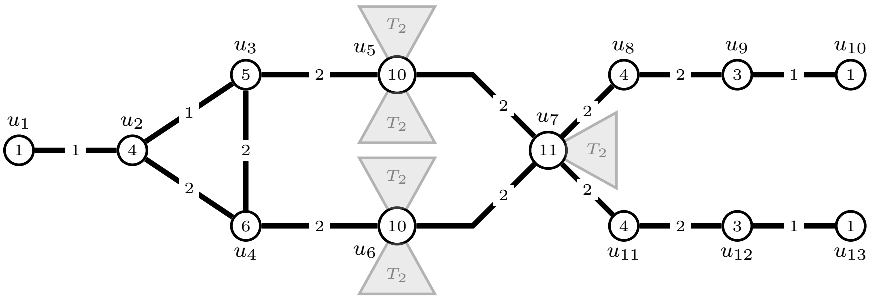
1. $\{\ell(u_1 u_2), \ell(u_1 u_3)\} = \{1, 2\}$,
2. $9 \in \{c_\ell(u_2), c_\ell(u_3)\}$,
3. *the one of $c_\ell(u_2)$ and $c_\ell(u_3)$ different from 9 can be any of 8 and 10.*

Proof. Let ℓ be a 2-labeling of T_2. So that $c_\ell(a_1) \neq c_\ell(a_2)$, we must have, say, $\ell(a_1 u_2) = 1$ and $\ell(a_2 u_2) = 2$. Note that whatever the label of $a_1 a_2$ is, no conflict involving a_1, a_2 and u_2 can arise, due to the larger degree of u_2. These arguments also apply around the b_i's,

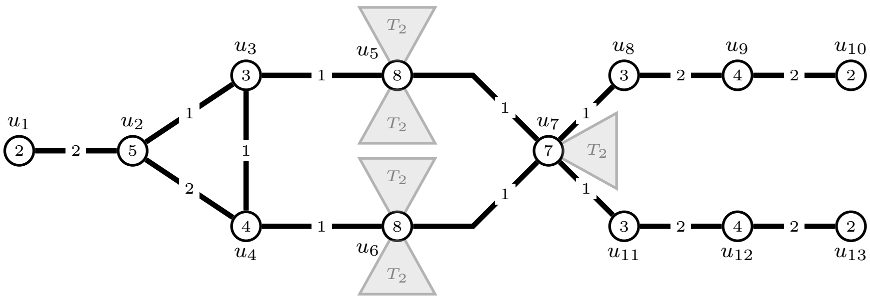


(a) $\{c_{\ell_1}(u_2), c_{\ell_1}(u_3)\} = \{8, 9\}$ (b) $\{c_{\ell_2}(u_2), c_{\ell_2}(u_3)\} = \{9, 10\}$

Figure 4: The two main 2-labelings ℓ_1 and ℓ_2 of T_2. An integer in a circle representing a vertex is the color of this vertex for the depicted 2-labeling.



(a) Input is labeled 1



(b) Input is labeled 2

Figure 5: 2-labelings of the spreading gadget $G^{\curlywedge}$. A triangle with "T_2" marked in indicates that a copy of the gadget T_2 is attached via its root vertex. That is, u_5 (resp., u_6) is identified to the roots of two copies of T_2, while u_7 is identified to the root of one copy of T_2. An integer in a circle representing a vertex is the color of this vertex for the depicted 2-labeling.

c_i's and d_i's. In particular, the labels of the four edges joining u_2 and the a_i's and b_i's bring 6 to the color of u_2, and similarly the labels of the four edges joining u_3 and the c_i's and d_i's bring 6 to the color of u_3.

Now, so that $c_\ell(u_2) \neq c_\ell(u_3)$, we must have, say, $\ell(u_1u_2) = 1$ and $\ell(u_1u_3) = 2$. Then no conflict involving u_2 and u_3 can arise, no matter whether u_2u_3 is labeled 1 or 2. In the first case, we get $(c_\ell(u_2), c_\ell(u_3)) = (8, 9)$, while we get $(c_\ell(u_2), c_\ell(u_3)) = (9, 10)$ in the second case. $\square$

We now introduce the *spreading gadget* $G^{\curlywedge}$, depicted in Figure 5. The edge $i(G^{\curlywedge}) = u_1u_2$ of $G^{\curlywedge}$ is its input, while its edges $o_1(G^{\curlywedge}) = u_9u_{10}$ and $o_2(G^{\curlywedge}) = u_{12}u_{13}$ are its outputs. Some properties of $G^{\curlywedge}$ are the following.

Lemma 4.14. *In every* 2*-labeling* ℓ *of* $G^{\curlywedge}$*, the input and the two outputs are assigned the same label, i.e.,* $\ell(u_1u_2) = \ell(u_9u_{10}) = \ell(u_{12}u_{13})$*. This label can be either of* 1 *and* 2*.*

Proof. Assume ℓ is a 2-labeling of $G^{\curlywedge}$. Note that we must have $\ell(u_3u_5) = \ell(u_4u_6)$. Indeed, suppose w.l.o.g. that $\ell(u_3u_5) = 1$ and $\ell(u_4u_6) = 2$. Since there are two copies of T_2 attached to u_5, by Lemma 4.13, the color of u_5 is $7+\ell(u_5u_7)$ and it is adjacent to a vertex with color 9 (in T_2). Similarly, because of the two copies of T_2 attached to u_6, the color of u_6 is $8 + \ell(u_6u_7)$ and it is adjacent to a vertex with color 9 (in T_2). Then we must have $\ell(u_5u_7) = 1$ and $\ell(u_6u_7) = 2$, so that $c_\ell(u_5) = 8$ and $c_\ell(u_6) = 10$. We also know that a neighbor of u_7 from the graph T_2 attached to it has color 9, and that this graph T_2 provides 3 to the color of u_7 by Lemma 4.13. Then, u_7 has color $6 + \ell(u_7u_8) + \ell(u_7u_{11})$, and the two edges u_7u_8 and u_7u_{11} must be labeled (with 1 or 2) in such a way that the color of u_7 does not meet any value in $\{8, 9, 10\}$, which is impossible.

On the contrary, there exists a 2-labeling ℓ such that $\ell(u_3u_5) = \ell(u_4u_6) = 1$. Because of the arguments above, we have $\ell(u_5u_7) = \ell(u_6u_7) = 1$ and $c_\ell(u_5) = c_\ell(u_6) = 8$. Recall that we may assume that the labeling of the graph T_2 attached to u_7 is such that the two vertices that are adjacent with u_7 have color 9 and 8 (Lemma 4.13). Besides, the labeling of this graph T_2 provides 3 to the color of u_7. Thus, the color of u_7 is at least 5, and the edges u_7u_8 and u_7u_{11} are labeled in such a way that the color of u_7 is not 9 or 8. The only possibility is to have $\ell(u_7u_8) = \ell(u_7u_{11}) = 1$ since, in this situation, we get $c_\ell(u_7) = 7$. It can be checked that, by similar arguments, there exists a 2-labeling ℓ such that $\ell(u_3u_5) = \ell(u_4u_6) = 2$.

Now suppose $\ell(u_1u_2) = 1$, and consider the edges u_2u_3 and u_2u_4 (see Figure 5 (a) for an illustration). First, if $\ell(u_2u_3) = \ell(u_2u_4)$, then note that ℓ is not a proper labeling according to the arguments above since we would necessarily have $\ell(u_3u_5) \neq \ell(u_4u_6)$ so that $c_\ell(u_3) \neq c_\ell(u_4)$. Thus, $\ell(u_2u_3) = 1$ and $\ell(u_2u_4) = 2$ without loss of generality, and $c_\ell(u_2) = 4$. Note that, if $\ell(u_3u_4) = 1$, then we necessarily get that $c_\ell(u_3)$ or $c_\ell(u_4)$ is equal to $c_\ell(u_2)$ since we need $\ell(u_3u_5) = \ell(u_4u_6)$. Thus $\ell(u_3u_4) = 2$. We then have $\ell(u_3u_5) = 2$ so that $c_\ell(u_3) \neq c_\ell(u_2)$, and also $\ell(u_4u_6) = 2$ so that $c_\ell(u_4) \neq c_\ell(u_3)$ (and because $\ell(u_4u_6) = \ell(u_3u_5)$ by arguments above).

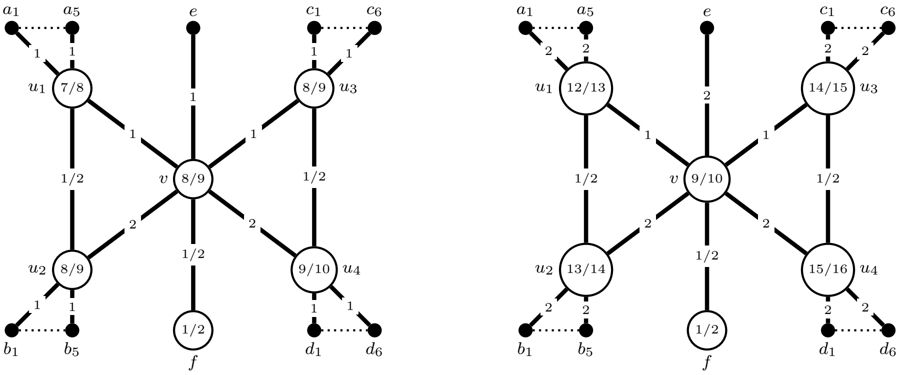
According to the arguments above, we have $\ell(u_3u_5) = \ell(u_4u_6) = 2$ and $\ell(u_7u_8) = \ell(u_7u_{11}) = 2$ under the assumption $\ell(u_1u_2) = 1$. Then $\ell(u_9u_{10}) = \ell(u_{12}u_{13}) = 1$ to avoid conflicts. Thus, assuming the input of $G^{\curlywedge}$ is labeled 1, also its two outputs are labeled 1.

A similar case analysis yields an analogous conclusion when $\ell(u_1u_2) = 2$, see Figure 5 (b). Let us point out that, in both cases, the label of the edges u_8u_9 and $u_{11}u_{12}$ could be any of 1 and 2 at this point. □

In what follows, we will combine copies of $G^{\curlywedge}$ via their inputs and outputs. Let us first prove that this preserves the labeling properties of Lemma 4.14.

Lemma 4.15. *Let* G_1 *and* G_2 *be two copies of* $G^{\curlywedge}$*, and let* G *be the graph obtained by identifying* $o_1(G_1)$ *and* $i(G_2)$*. Then, in every* 2*-labeling of* G*, all of* $i(G_1)$*,* $o_1(G_1) = i(G_2)$*,* $o_2(G_1)$*,* $o_1(G_2)$ *and* $o_2(G_2)$ *are assigned the same label. This label can be either of* 1 *and* 2*.*

Proof. Let ℓ be a 2-labeling of G. Because $i(G^{\curlywedge})$ and $o_1(G^{\curlywedge})$ are pendant edges of $G^{\curlywedge}$, in G the combination of G_1 and G_2 does not grant new labeling possibilities (Indeed, note that in proofs of Lemmas 4.13 and 4.14, we never considered the color of vertex u_1 nor the ones of vertices u_{10} and u_{13}, so changing their degree will not impact the properties of any labeling $G^{\curlywedge}$). In other words, ℓ, when restricted to G_1 and G_2, is a 2-labeling, which thus verifies the properties in Lemma 4.14. From this, and because the input of G_2 and



(a) All inputs are labeled 1

(b) All inputs are labeled 2

Figure 6: The two main 2-labelings of the 2-forcing graph F. An integer in a circle representing a vertex is the color of this vertex for the depicted 2-labeling.

an output of G_1 coincide, we directly get that all of $i(G_1)$, $o_1(G_1) = i(G_2)$, $o_2(G_1)$, $o_1(G_2)$ and $o_2(G_2)$ must receive the same label by ℓ. It just remains to show that ℓ can indeed be adjusted so that no conflict arises around the "connection points". That is, we must make sure that the colors of vertices u_9 and u_{10} in G_1 (which correspond to u_1 and u_2 in G_2) are not equal.

Assume first the label assigned by ℓ to the input and outputs of G is 1. If we just consider as ℓ the edge-labeling described in the proof of Lemma 4.14 (depicted in Figure 5 (a)), then, in G_1, we have $u_9u_{10} = 1$ and u_8u_9 can be freely chosen to be labeled 2, as pointed out in the proof. This leads vertex u_9 of G_1 to have color 3. Now, by how ℓ propagates in G_2 (assuming its input edge is labeled 1), its vertex u_2, which is u_{10} of G_1, gets color 4. So there is no conflict involving vertices u_9 and u_{10} of G_1, around the edge where the identification was performed.

Through similar arguments, it can be checked that when the input and outputs of G are assigned label 2 by ℓ, then ℓ can flow through G_1 and G_2 so that, in G_1, we have $\ell(u_7u_8) = \ell(u_7u_{11}) = 1$ and $\ell(u_8u_9) = \ell(u_{11}u_{12}) = \ell(u_9u_{10}) = \ell(u_{12}u_{13}) = 2$ (see Figure 5 (b)). That way, the color of u_9 in G_1 is 4 while the color of u_{10} (which is u_2 in G_2) is 5, and there is no conflict. $\square$

According to Lemma 4.15, starting from copies of $G^{\curlywedge}$ and concatenating them through their input and outputs, we can now obtain a *generator graph* $G_{1/2}$ having one input $i(G_{1/2})$ and arbitrarily many outputs $o_1(G_{1/2}), o_2(G_{1/2}), \dots$ such that any 2-labeling of $G_{1/2}$ is such that the input can be labeled any of 1 and 2, but all outputs have the same label as the input. In what follows, we introduce some more structure to the generator graph to force its input to be labeled 2 by any 2-labeling.

The 2-*forcing graph* F is the graph depicted in Figure 6. It has twenty-three inputs ev, $a_1u_1, \dots, a_5u_1$, $b_1u_2, \dots, b_5u_2$, $c_1u_3, \dots, c_6u_3$ and $d_1u_4, \dots, d_6u_4$ (and no output). Its main labeling properties are the following:

Lemma 4.16. *Assume ℓ is a 2-labeling of the 2-forcing gadget F where all inputs are assigned the same label. Then all inputs must be labeled 2.*

Proof. Assume the twenty-three inputs of F are labeled 1 (see Figure 6 (a)). This brings 5 to both the color of u_1 and u_2. So that $c_\ell(u_1) \neq c_\ell(u_2)$, we must thus have, say, $\ell(u_1v) = 1$ and $\ell(u_2v) = 2$, which implies that, regardless of $\ell(u_1u_2)$, we must have $8 \in \{c_\ell(u_1), c_\ell(u_2)\}$. The same arguments for u_3, u_4 and the twelve inputs connected to them imply that we must have, say, $\ell(u_3v) = 1$, $\ell(u_4v) = 2$, and $9 \in \{c_\ell(u_3), c_\ell(u_4)\}$. Now, since $\ell(ev) = 1$ by assumption, we note that we must have $c_\ell(v) = 8$ or $c_\ell(v) = 9$, depending on whether $\ell(vf) = 1$ or $\ell(vf) = 2$. Vertex v is then involved in a conflict, a contradiction.

On the other hand, there exist 2-labelings of F where all inputs are labeled 2. An example is given in Figure 6 (b). □

Now take the generator graph $G_{1/2}$, choose twenty-three of its outputs (as mentioned earlier, we can assume $G_{1/2}$ has arbitrarily many outputs), and identify these with the twenty-three inputs of a copy of the 2-forcing gadget F. We call the resulting graph the 2-*generator graph* G_2. The input of G_2 is the input of $G_{1/2}$, and the outputs of G_2 are the outputs of $G_{1/2}$ that are different from the twenty-three outputs used for the connection to the copy of F. Since $G_{1/2}$ can have arbitrarily many outputs, so does G_2.

Lemma 4.17. *Assume ℓ is a 2-labeling of the 2-generator graph G_2. Then the input and all outputs of G_2 must be labeled 2.*

Proof. As described earlier, the input and all outputs of the generator graph $G_{1/2}$ used to construct G_2 must be assigned the same label by a 2-labeling ℓ of G_2 (in particular, this is not impacted by the connection to the 2-forcing gadget F). This label cannot be 1, as otherwise ℓ could not be propagated through the 2-forcing gadget F in G_2, by Lemma 4.16. Thus, this label must be 2.

Furthermore, there do exist 2-labelings of G_2 where the input and all outputs are labeled 2. First of all, recall that both the generator gadget $G_{1/2}$ in G_2 (by Lemma 4.15) and the 2-forcing gadget F in G_2 (by Lemma 4.16) admit such. Now we just need to show that 2-labelings of $G_{1/2}$ and F where the inputs and outputs are labeled 2 can indeed be combined to one of G_2 in such a way that no conflict arises. For that, we just need to make sure that a vertex u_9 (or u_{12}) being part of a copy of $G^{\curlywedge}$ in $G_{1/2}$ is not in conflict with a resulting neighbor in the used copy of F. Since u_9 in $G^{\curlywedge}$ has degree 2 and the outputs are assigned label 2, the color of that u_9 is either 3 or 4. In a copy of F, such a vertex u_9 of $G^{\curlywedge}$ is adjacent to either of vertices u_1, u_2, v, u_3 or u_4 in the copy of F. We note that each of these vertices has degree at least 6, and thus its color is at least 6. So no conflict can arise when combining 2-labelings of $G_{1/2}$ and F in G_2. □

We are now ready for our conclusion.

Theorem 4.18. *For every $\Delta \geq 16$, there exists a graph G with maximum degree Δ verifying $mS_2(G) = 2\Delta$ and $mS_3(G) = \Delta$.*

Proof. Let $\Delta \geq 16$ be fixed, and consider the 2-generator G_2 with Δ outputs. Then let G be the graph obtained from G_2 by identifying the vertices with degree 1 of these Δ outputs to a single vertex v^*. Note that $\Delta(G_2) = 8$ (the largest degree being attained for vertices u_2 and u_3 of the forcing gadget F), so we have $\Delta(G) = d(v^*) = \Delta$.

Now consider a 2-labeling of G. Graph G contains G_2, and, by Lemma 4.17, all input and outputs must be labeled 2. In particular, all edges incident to v^* must be labeled 2, which means that $c_\ell(v^*) = 2\Delta$. We note that there actually exist such 2-labelings of G, since G_2 admits some (as pointed out in the proof of Lemma 4.17), and the only conflicts that can arise are between v^* and its neighbors. These neighbors are actually vertices u_9 or u_{12} of copies of $G^{\curlywedge}$ in G_2, and are thus of degree 2 while v^* has degree at least 16. So these vertices cannot be in conflict.

We now claim that we can produce a 3-labeling of G where the maximum color is Δ. To see this holds, start with just using $1, 2$ as above, getting an initial labeling ℓ. Since v^* results from the identification of outputs of G_2, thus of outputs of $G^{\curlywedge}$, the neighbors of

v^* are all some vertices either u_9 or u_{12} from some copies of $G^{\curlywedge}$, each such vertex being adjacent to some vertex u_8 or u_{11} from the same copy of $G^{\curlywedge}$, each such vertex being also adjacent to some vertex u_7 in that copy. We modify ℓ by considering every output incident to v^*, and modifying the label of the associated edges u_7u_8 and u_7u_{11} (in the corresponding copy of $G^{\curlywedge}$) to 3, and the label of the associated edges u_8u_9, $u_{11}u_{12}$, u_9v^* and $u_{12}v^*$ to 1. This raises no conflict, as, in every incident copy of $G^{\curlywedge}$, the corresponding u_7 gets color 11, vertices u_8 and u_{11} get color 4, and vertices u_9 and u_{12} get color 2. Once this modification has been applied to every output incident to v^*, all its incident edges are assigned label 1, so $c_\ell(v^*) = \Delta$ (which is so large that it cannot be the color of a neighbor of v^*, since all neighbors of v^* have degree 2). Furthermore, this is the largest color, since:

- only vertices u_7, u_8 and u_{11} from some copy of $G^{\curlywedge}$ are incident to edges labeled 3, and, in their case, their color is less than Δ (as pointed out above); and
- all other vertices of G different from v^* have degree at most 8 and are not incident to edges labeled 3; so the color of these vertices is at most $\Delta \geq 16$.

So $mS_3(G) = \Delta$, while $mS_2(G) = 2\Delta$. □

5. Conclusions and perspectives

In this work, we have investigated the minimum maximum color $mS_k(G)$ that one can generate by a k-labeling of a given graph G. This parameter is related to the well-known 1-2-3 Conjecture, and we have thus mainly focused on classes of graphs for which the parameter χ_Σ is relatively well understood (complete graphs, complete bipartite graphs, trees, bipartite graphs). We have provided bounds on mS_k for such graphs, some of which are tight. An interesting aspect to us was the algorithmic complexity of determining the value of $mS_k(G)$ for a given graph G. We have shown that the complexity of this problem is highly dependent of the input graph. As a consequence, that problem is sometimes hard (NP-complete) or easy (polynomial-time solvable). The proof we have provided that this problem is easy for some graph classes is a consequence of a more general polynomial-time algorithm we have designed for graphs with bounded treewidth, which has more general side consequences on algorithmic aspects related to the 1-2-3 Conjecture. Finally, we have also investigated the trade-off between using larger labels and aiming at generating smaller colors.

We leave open a number of questions, however, and we think that they could lead to further work on the topic. In particular:

- Claim 2.2 states that for every nice graph G with maximum degree Δ, the value of $mS_2(G)$ lies in between Δ and 2Δ. In Section 4.1.1, we have shown that there exist bipartite graphs G with maximum degree $\Delta \in \{2, 3\}$ for which $mS_2(G)$ reaches the upper bound 2Δ. We do not know whether this upper bound is also correct for larger value of Δ. So we ask: For every $\Delta \geq 4$, are there bipartite graphs G with maximum degree Δ such that $mS_2(G) = 2\Delta$?
- In Section 4.2, we have essentially shown that, for a nice tree T with maximum degree Δ, the value of $mS_2(T)$ is one of $\Delta, \Delta+1, \Delta+2$. Furthermore, our algorithm from Section 3.2 attests that $mS_2(T)$ can be determined in polynomial time. We are not sure, however, about a nice characterization, for instance in terms of particular subtrees, of the trees T with $mS_2(T)$ being Δ, $\Delta+1$ or $\Delta+2$.

- A similar concern applies to odd multi-cacti. We have essentially shown, in Section 4.1.2, that for an odd multi-cactus G with maximum degree $\Delta \geq 3$ the value of $mS_3(G)$ is either $\Delta + 1$ or $\Delta + 2$. Is there a nice characterization of the odd multi-cacti for which mS_3 is either of these two values?

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